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Safety and Throughput Analysis of Automated Highway Systems

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Abstract

We investigate the effect of a number of design alternatives on the safety and capacity of an Automated Highway System. Our methodology makes use of two computational tools, designed to highlight the fundamental limitations of the vehicle dynamics, sensing and control strategies and inter-vehicle communication. The first tool produces the minimum spacing necessary for two vehicles not to collide, as a function of their state and capabilities. The second tool investigates the multiple collisions that may occur in a string of vehicles if the spacing requirements of the first tool are violated. The example we have in mind is the emergency deceleration of a platoon of vehicles. We use the tools to establish limits on the safety and throughput that can be expected if different automated highway concepts are implemented.

Our results indicate that on an Automated Highway System that supports platooning, severe intra-platoon collisions may occur under emergency situations. We analyze the effect of the collisions on both safety and throughput and investigate the sensitivity of these quantities to a number of design parameters. We identify as the most important parameters inter-vehicle coordination, on-line braking capability estimation and platoon organization in terms of braking capability.

1 Introduction

Highway congestion is an ever increasing problem, especially in and around urban areas. More than 50% of the congestion can be attributed to non-recurrent causes such as accidents, due primarily to driver fatigue, inattentiveness and aggressiveness. One of the promising solutions suggested for this problem is traffic automation, partial or full. The substitution of computer control for human control may reduce or eliminate “driver errors” and hence improve safety. Moreover, as the automated controller can react to disturbances faster than a human driver, automation may also decrease the average inter-vehicle spacing and hence increase throughput and reduce congestion and delays. It is hoped that this improvement can be achieved with a relatively small investment on infrastructure, compared to the investment needed for building new highways, especially in urban areas where real estate is expensive.

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The design of a fully *Automated Highway System* (AHS) is an extremely challenging control problem. The dominant characteristic of an AHS is that a large number of vehicles, equipped with sensing, communication and control capabilities, are trying to make efficient use of a congested, common resource (the highway). Unless special measures are taken, the optimum policy for each vehicle (reduce travel time) may not coincide with the overall system goals (increase safety and throughput). A number of alternatives have been proposed for organizing traffic on an AHS in an attempt to achieve an acceptable compromise. The effectiveness of a proposed AHS design should be judged by performance metrics in the areas of safety, throughput, fuel efficiency, environmental impact, vehicle and infrastructure cost, social fairness, etc. In this paper, we restrict our attention to *safety* and *throughput* and attempt to analyze the effect of a number of design choices on these two quantities. Two computational tools are developed for this purpose. The tools are based on *analytical calculations* that explore the fundamental limitations of the vehicle dynamics, the sensing and control capabilities and the inter-vehicle information exchange. The tools do not require detailed information about the design of the vehicle controllers and can therefore be very useful at an early stage in the design process (the concept definition phase) to assess relative merits of different AHS attributes.

The contribution of the work presented here is threefold. First, our work establishes a *formal methodology* for performing safety and throughput studies for AHS. Second, we provide software tools that allow us to apply this methodology to situations that are relevant in practice. Finally, the results we present for specific AHS design concepts (such as platooning) are important in themselves, as they highlight the limitations of these concepts. Our tools and analysis methodology fall in the range between theoretical investigation, and simulation and experimental studies. The algorithmic nature of the tools allows us to extract information for situations that are too complicated to study theoretically. On the other hand, our tools rely on simple models for the AHS design and can therefore be used even with conceptual or partial designs, situations where simulation and experimentation are inapplicable¹. Of course the simplicity of the models also implies that our techniques can only produce *estimates* for safety and throughput. Our results can be viewed as an attempt to establish *performance limits* of an AHS concept; how close a particular implementation gets to these limits will have to be determined by simulation and experiments on the detailed AHS design. Even though in this paper we concentrate on a fully automated AHS, our analysis methodology can be extended to mixed and manual traffic, provided appropriate models for human driver behavior are made available.

The paper is arranged in seven sections. In Section 2 we provide an explicit problem formulation and introduce the computational tools and analysis methodology. We also summarize the results of our analysis (presented in detail in subsequent sections). In Section 3 the tool design is discussed in greater detail. We provide some theoretical background to justify the design of each tool and briefly discuss the coding process and other possible tool uses not addressed in this paper. In Sections 4 to 6 we present the safety–throughput case study. In Section 4 we summarize “pipeline” throughput results obtained using one of the tools and in Section 5 collision statistics collected using the other. In Section 6 we discuss how the tools can be used in tandem to estimate the effect of collisions and their propagation on throughput. Finally, in Section 7 we summarize our findings and provide directions for further investigation. More results and modifications to accommodate different vehicle models are given in the appendix.

¹It should also be noted here that simulation and experimentation alone may be insufficient for safety analysis, as the probability of missing an unsafe situation for a well designed system may still be fairly high.

2 Problem Formulation and Overview of Results

2.1 Objectives

We concentrate on the effects of longitudinal vehicle control on AHS safety and throughput. Our analysis relies on two fundamental assumptions: *all vehicles are automatically controlled* and *lateral operation is perfect*, even in the presence of collisions. We hope to be able to relax the former assumption without major changes in our analysis methodology and tools. For detailed studies of the latter assumption the reader is referred to [1, 2, 3, 4, 5].

Calculating the exact throughput of an AHS is a complicated problem [6, 7, 8]. Throughput depends on the locations and demands of entrances and exits [9] and lane change patterns [10]. Our throughput calculations ignore these effects. We consider an *AHS pipe*, a single lane AHS without entrances and exits and estimate the maximum possible steady state flow in vehicles per lane per hour through such a pipe. Following the terminology introduced in [11], we call this the *AHS pipeline capacity*. The per lane flow in an actual multi-lane AHS will typically be less than the pipeline capacity due to lane changing, merging, entrances and exits. Pipeline capacity can therefore be viewed as an upper bound on the flow that can be achieved for a particular AHS design. Under the assumed steady state conditions the pipeline capacity is uniquely determined by the relationship between speed and inter-vehicle spacing.

Highway safety is typically characterized by the number and severity of collisions. The top four causes of collisions on U.S. roadways, namely rear-end, single vehicle roadway departure, lane change/merge, and intersection collision, account for over 90% of total collisions and injuries [12]. The assumption of perfect lateral control allows us to ignore single vehicle roadway departures. The intersection collisions may also be ignored as we assume that automated travel will only be implemented on limited access freeways. The occurrence of collisions in the remaining categories directly depends on the AHS speed-spacing relationship and the lane change logic.

The above discussion indicates that two crucial elements for both safety and throughput are the speed-spacing relation and lane change policies (including entry, exit and merging in the latter). Here we deal mostly with the effect of the speed-spacing policy². On regular highways the human driver is responsible for selecting the speed-spacing relationship; on an AHS, this relationship is set by the AHS designer. Two different types of AHS spacing policies are considered in this paper: *individual vehicle* and *platooning*. In an individual vehicle AHS, each vehicle determines its following distance based on its own control capabilities, operating speed and possible information about the preceding vehicle. In a platoon based AHS, on the other hand, vehicles travel in closely spaced groups (*platoons*) of up to 20 vehicles, with constant intra-platoon separations of the order of 1 – 5m. Platoons are isolated from each other by larger distances, so as to avoid inter-platoon collisions. The close packing of vehicles can yield up to 3-4 times increase in highway capacity [14]; furthermore, it is believed that this can be done without an adverse effect on safety. It is conjectured that even in case of a failure within a platoon, collisions will happen at low relative velocity (because of the close following) resulting in minor damage and injuries [15]. Here we investigate both the safety and the throughput claims.

2.2 Tool Development

A number of controllers have been proposed in the literature for longitudinal control of individual automated vehicles, platoon leaders, platoon followers, lane change maneuvers and platoon forma-

²The use of our techniques for determining safe lane changing policies is illustrated in [13]. For an application of our methodology to a detailed throughput study (including the effects of lane changing) the reader is referred to [8].

tion/dissipation maneuvers [16, 17, 18, 19, 13, 20]. In [13] it was shown how, under the perfect lateral operation assumption, controllers for individual vehicles can be designed to guarantee no collisions. It was then shown how, using an appropriate coordination strategy, the results can be extended to an AHS that supports platooning, under the additional assumption of *normal operation* (no malfunctions or adverse environmental conditions).

Here, based on the results of [13], we develop a *spacing tool* to produce requirements on the minimum inter-individual-vehicle and inter-platoon spacing needed for safety and convert this information to estimates of pipeline capacity. The spacing tool accepts as input the velocities and accelerations of two vehicles and their deceleration capabilities and returns the minimum spacing needed to guarantee that the two vehicles do not collide. The development of the tool directly draws on the theoretical results of [13]. The tool output indicates that the safe inter-vehicle spacing is particularly sensitive to *differences in the braking capability and the initial velocities of the vehicles*. Given a probability distribution for deceleration capabilities, a straight forward calculation (Section 4) relates the output of the spacing tool to expected pipeline capacity.

In a string of vehicles where the safe spacing requirements are violated (as is typically the case within a platoon) multiple collisions are possible. The platooning safety theorem of [13] is based on the assumption that these collisions will not be severe. Here we examine this assumption further for the case of emergency platoon deceleration, using a *collision tool*. The collision tool accepts as input the initial configuration of a string of vehicles, assumes a simple deceleration strategy and returns the severity of all collisions that take place among the vehicles of the string, until they all come to a stop. Assuming the deceleration capability distribution used in the throughput calculations, the tool can be used to obtain collision statistics such as average number of collisions and expected relative velocities.

Finally, we use the two tools in conjunction to establish a link between possible intra-platoon collisions and pipeline capacity of platoon based AHS.

2.3 Overview of the Results

2.3.1 Normal Mode Pipeline Throughput

In all the cases considered here, the AHS throughput predicted by the tools turns out to be substantially greater than the one observed for manual driving, indicating that major benefits may be obtained by automation. We attempt to estimate the sensitivity of these results with respect to a number of design parameters. In particular we consider variations in:

1. nominal speed of operation,
2. platoon size,
3. availability of information about deceleration capabilities, and
4. introduction of minimum headway requirements.

Our results indicate that the effect of the last element is rather small. As expected, throughput decreases as the minimum required headway increases and the reduction is more pronounced for smaller platoons. Throughput seems to be very sensitive to changes in the speed of operation, v , and platoon size, N . To capture the effect of speed variations, data was collected over the range $v \in [0, 40]m/s$. In all cases, the *throughput starts at zero when the speed is zero, increases to a peak* (typically in the above range) *and rolls off at high speeds*³. To capture the effect of platoon

³This observation is in accordance with earlier studies [11] and data collected for manual driving [21].

size data was collected for $N = 1, 3, 5$ and 7 . In most cases *an increase in platoon size leads to a dramatic improvement in throughput*.

Throughput values are also very sensitive to information about the deceleration capabilities of the vehicles. We investigate the following three scenarios:

1. vehicles have access to their own deceleration capability and that of the vehicle ahead,
2. vehicles have access only to their own deceleration capability,
3. vehicles have access to neither capability.

For the last two cases we assume that, in order to maintain a safe spacing, vehicles assume the worst possible value allowable by the deceleration capability distribution for the missing pieces of information. Our results indicate that the *throughput increases dramatically as the amount of information increases*. An interesting fact is that when both deceleration capabilities are available the throughput values for platoon sizes up to $N = 7$ are comparable. The throughput obtained in this case is almost an order of magnitude higher than that of manual driving.

2.3.2 Collisions Due to Emergency Braking

The throughput values discussed above were obtained under the assumption that the operation of the system is normal. Under these conditions, controllers can be designed to guarantee that no inter- or intra-platoon collisions will occur [13, 17]. If the normal operation assumption is violated however, collisions may be unavoidable. In this paper we restrict our attention on failures that give rise to emergency deceleration⁴. The string stability results of [17] indicate that coordination of braking effort among the vehicles of a platoon can ensure collision free operation. In the absence of coordination, however, the results of [13] indicate that the typical intra-platoon spacing is insufficient to guarantee collision free operation in the presence of mismatch in deceleration capabilities and/or delays. If a fault (e.g., a brakes on failure of a vehicle or hard deceleration by a platoon leader to avoid an obstacle) renders coordination impossible, multiple intra-platoon collisions may be observed.

To investigate this phenomenon the collision tool was used to collect statistics of collisions per incident of emergency (uncoordinated) braking. The statistics we are primarily interested in relate to the relative velocities at impact (a measure of collision severity according to [23]) and the average number of collisions. The tool runs indicate that the results are very sensitive to the size, N , of the platoons considered. *For small platoons ($N = 2, 3$) collisions occur primarily at low relative velocities*, mostly below the $3ms^{-1}$ safety threshold quoted in [23]⁵. However, both the severity and the average number of collisions increases dramatically as N increases. *For $N = 5$ and 7 the probability of collisions at relative velocities higher than $3ms^{-1}$ is in fact substantial*. Further experiments were carried out to investigate sensitivity with respect to:

1. communication architecture within a platoon (hop-by-hop versus broadcast),
2. communication delay,
3. nominal speed of operation,
4. nominal intra-platoon spacing,
5. coefficient of restitution for the collisions,

⁴For a comprehensive list of faults and response strategies the reader is referred to [22].

⁵For $N = 1$ the system can be designed so that no collisions ever occur for the classes of faults considered here.

6. deceleration capability distribution.

Our results indicate that the collision statistics are relatively insensitive with respect to the first three parameters. They are however quite sensitive with respect to the remaining three. In particular, *the average severity of collision increases substantially as the intra-platoon spacing increases, as the collisions become more elastic and as the width of the deceleration distribution increases.* It should be stressed that the above statements concern the severity of collision statistics *averaged over the deceleration distribution.* The results do not necessarily hold for specific runs.

2.3.3 Relating Collisions and Throughput

The above discussion concerns only the intra-platoon collisions that occur per emergency braking incident. If we can guarantee that no inter-platoon collisions occur, then collisions do not propagate from one platoon to the next and the cases discussed above are all one needs to worry about. It is possible to guarantee absence of inter-platoon collisions for the class of faults considered here by allowing extra inter-platoon spacing. This can be done by using the two tools in conjunction. The collision tool is used to determine the worst collision that the first and the last vehicle of a platoon of given size can experience. The relative velocities of these collisions are then used to appropriately set the input of the spacing tool, to produce the minimum spacing such that, even if two adjacent platoons experience collisions due to emergencies, the collisions will not propagate from one to the other. This spacing is bound to be larger than the one in the absence of collisions and therefore the resulting throughput will be smaller.

Numerical experiments were conducted to quantify this observation. The results indicate that *the requirement that collisions do not propagate between platoons severely limits the throughput benefits from platooning.* The throughput obtained for platoons of size $N = 1$ up to size $N = 7$ are in fact comparable for a wide range of design parameters: operating speeds, communication architectures, intra-platoon spacing, deceleration distribution, coefficient of restitution, etc. It should be stressed that *the value of these experiments is more qualitative than quantitative,* as their accuracy is limited. For example, only one collision per lead/trail vehicle is considered here, whereas the runs of the collision tool indicate that these vehicles may experience multiple collisions, especially for larger platoons. Despite such biases, the results indicate that *the improvement in throughput by the formation of larger platoons is definitely not as dramatic as it seems if collisions are not considered; comparable throughput can be obtained without any collisions using individual vehicles.*

2.4 Related Work

The spacing tool was used in [11] to evaluate spacings and capacities of AHS concepts that differ in their hardware capabilities and separation policies. The work of [11] did not consider the effect of intra-platoon collisions on pipeline capacity thereby resulting in optimistic throughput values for a platoon based AHS. On the other hand, the work of [11] considered the effect of different degrees of inter-vehicle cooperation and different classes of vehicles. In this paper we will not explore these dimensions. In particular, we assume that all vehicles on the AHS are light duty passenger vehicles.

The problem of intra-platoon collisions and their dependence on system capabilities (control loop delays, braking capabilities) has been the subject of [24, 25]. The analysis in [24, 25] was restricted to the first collision in an incident; here we evaluate the effect of all intra-platoon collisions for a given incident. In [24] the safety of AHS was also compared with manual driving safety; this will not be attempted in this paper.

Here the intra-platoon collisions are evaluated in a probabilistic setting. Given probabilistic descriptions of the system capability, the collision tool calculates the average probability and

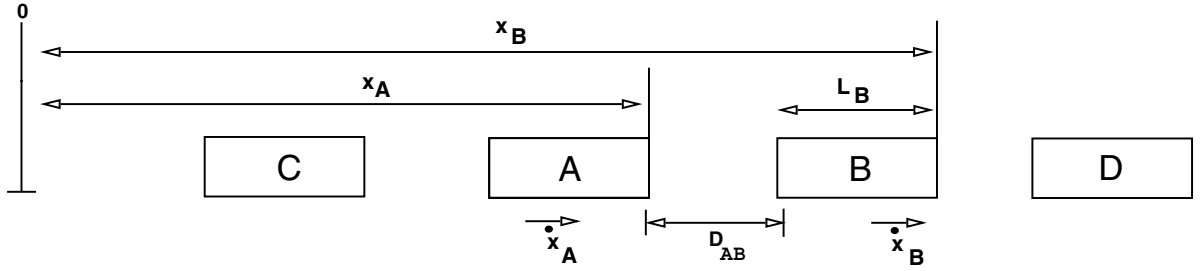


Figure 1: Vehicle Following

severity of collisions due to an emergency deceleration of the platoon. In a related paper, Lygeros and Lynch [26] determine deterministic necessary and sufficient conditions on the system capability parameters so that severe collisions due to emergency deceleration can be completely avoided.

The collision model used in this paper is rather simple and does not take into account effects such as bumper dynamics and turning moments generated due to off-centered collisions. Even more severe collisions might occur if a vehicle involved in a longitudinal collision departs its lane and either runs off the road or collides with vehicles in the adjacent lane. In this paper, we do not consider lateral effects of collisions. For detailed collision modeling, the reader is referred to [2, 3, 4, 5, 27].

3 Tool Development

3.1 Vehicle Model

Consider four vehicles (labeled A, B, C and D) moving along a single lane highway (Figure 1). Assume that vehicles A and B have lengths L_A and L_B and let x_A and x_B denote their positions with respect to a fixed reference on the road, \dot{x}_A, \dot{x}_B their longitudinal velocities and \ddot{x}_A, \ddot{x}_B their longitudinal accelerations. Assume that $x_D > x_B > x_A > x_C > 0$. We are primarily interested in the interaction between vehicles A and B, vehicles C and D will be used only in certain cases, to isolate the pair A-B from the rest of the highway. Following [13], assume that vehicle A can be modeled by a third order system and that the acceleration of vehicle B can not be measured by vehicle A. If we let $D_{AB} = x_B - x_A - L_B$ the pair A-B can be described by the state vector $x = [x_1 \ x_2 \ x_3 \ x_4]^T = [\dot{x}_A \ \ddot{x}_A \ D_{AB} \ \dot{D}_{AB}]^T$. After feedback linearization [28] the evolution of the state is given by the differential equation:

$$\frac{dx}{dt}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \ddot{x}_B(t) \quad (1)$$

where u is the jerk applied by the controller of vehicle A. Let $x(0) = x^0$ denote the state of the system at time $t = 0$. The vehicle dynamics are constrained by the engine, tire and road conditions. More specifically it is required that for all $t \geq 0$:

$$\begin{aligned} x(t) &\in \{x \in \mathbb{R}^4 | x_1 \in [v_A^{min}, v_A^{max}], x_2 \in [a_A^{min}, a_A^{max}], x_4 + x_1 \in [v_B^{min}, v_B^{max}]\} \\ u(t) &\in [j_A^{min}, j_A^{max}] \\ \ddot{x}_B(t) &\in [a_B^{min}, a_B^{max}] \end{aligned}$$

During normal highway operation it is assumed that vehicles will not be allowed to go backwards, therefore $v_A^{min} = v_B^{min} = 0$ will be used. v_A^{max} , v_B^{max} , a_A^{max} , a_B^{max} and j_A^{max} are imposed by engine limitations and play no role in the safety calculations. The values of the remaining bounds will be parameters for the safety tools.

If vehicle B happens to collide with vehicle D or vehicle C happens to collide with vehicle A, the state of the system (in particular the velocities of the vehicles involved in the collision) will undergo an almost instantaneous jump. If the change in velocity induced by the above two kinds of collisions are δv_B and δv_C and the collisions take place at times T_B and T_C respectively, then:

$$\begin{aligned} x_4(T_B^+) &= x_4(T_B^-) + \delta v_B \\ x_1(T_C^+) &= x_1(T_C^-) - \delta v_C \\ x_4(T_C^+) &= x_4(T_C^-) + \delta v_C \end{aligned}$$

where T_B^- and T_B^+ denote the time right before and right after the collision of vehicle B (similarly for C). In the coordinate system considered here, $\delta v_B \leq 0$ and $\delta v_C \leq 0$. Assume that vehicle B can hit vehicle D with relative velocity at most v_B and vehicle C can hit vehicle A with relative velocity at most v_C . If one collision of each kind takes place in the time interval of interest then the effect of vehicles B and C on vehicle A can be summarized as a disturbance:

$$\begin{aligned} d \in \{(\ddot{x}_B, (T_B, \delta v_B), (T_C, \delta v_C)) \mid \ddot{x}_B(t) \in [a_B^{min}, a_B^{max}] \text{ for all } t \geq 0, \\ 0 \leq T_B, \delta v_B \in [\max\{v_B, x_4(T_B) + x_1(T_B)\}, 0], \\ 0 \leq T_C, \delta v_C \in [v_C, 0]\} \end{aligned} \quad (2)$$

The complicated bound on δv_B is dictated by the fact that we require that $x_4(T_B^+) + x_1(T_B^+) \geq v_B^{min} = 0$. The above notation can also be used to model the situation where no collisions take place by simply setting $v_B = v_C = 0$.

This model is adequate for the case where there is no communication of state information between the platoons and no sensing and actuation delays. To compare the various AHS concepts other models may be needed. Appendix C contains different models that have been used in [11] to capture the effect of delays and inter-platoon communication.

3.2 Spacing Tool

For the purpose of safety we would like vehicles to avoid collisions whenever possible. For vehicle A, this requirement can be encoded by a cost function:

$$J(x^0, u, d) = -\inf_{t \geq 0} x_3(t) \quad (3)$$

If for a given initial condition x^0 and a given choice of u and d , $J(x^0, u, d) \leq 0$, vehicle A will not collide with vehicle B (it may still be hit by vehicle C). We would like vehicle A to remain safe in this sense whatever vehicles C and B choose to do. We therefore seek the worst possible strategy of vehicles B and C (d) and the best possible strategy of vehicle A (u). In other words we seek a pair (u^*, d^*) such that for all other u and d :

$$J(x^0, u^*, d) \leq J(x^0, u^*, d^*) \leq J(x^0, u, d^*) \quad (4)$$

If equation (4) holds for all x^0 , the pair (u^*, d^*) is called a *global saddle strategy* of the two player, zero sum game between u and d over J .

Our goal now is to find a global saddle strategy to the two player, zero sum game for the vehicle model described in the previous section. Consider the candidate:

$$u^*(t) = \begin{cases} j_A^{min} & \text{if } t \leq T_1 \\ 0 & \text{if } t > T_1 \end{cases} \quad (5)$$

$$d^*(t) = \{ \ddot{x}_B^*, (T_B^*, \delta v_B^*), (T_C^*, \delta v_C^*) \} \quad (6)$$

where $T_B^* = T_C^* = 0$, $\delta v_B^* = \max\{v_B, x_4^0 + x_1^0\}$, $\delta v_C^* = v_C$ and:

$$\ddot{x}_B^*(t) = \begin{cases} a_B^{min} & \text{if } t \leq T_2 \\ 0 & \text{if } t > T_2 \end{cases} \quad (7)$$

T_1 denotes the time when the acceleration of vehicle A reaches a_A^{min} under j_A^{min} and T_2 the time when vehicle B stops under a_B^{min} . The candidate saddle strategy simply requires both vehicles to decelerate as hard as possible and both collisions (B colliding with D and C colliding with A) to take place at time $t = 0$ with the maximum allowable relative velocity. In [13] it was shown that:

Lemma 1 (u^*, d^*) is a global saddle strategy for cost $J(x^0, u, d)$.

Let $x^*(t)$ be the state trajectory under (u^*, d^*) , let $J^*(x^0) = J(x^0, u^*, d^*)$ be the saddle cost and let T_3 be the time when vehicle A stops under u^* ($x_1^*(T_3) = 0$). Note that $x_3^*(t)$ is a continuously differentiable function of t defined on the compact interval $[0, T_3]$, with derivative $x_4^*(t)$. Therefore:

Lemma 2 There exists $\hat{T} \in [0, T_3]$ such that $J^*(x^0) = -x_3^*(\hat{T})$ and either $\hat{T} = 0$ or $\hat{T} = T_3$ or $x_4^*(\hat{T}) = 0$.

This lemma allows us to calculate $J^*(x^0)$ for a given initial condition x^0 . In particular it allows us to distinguish safe situations ($J^*(x^0) < 0$) from unsafe ones ($J^*(x^0) > 0$) and determine the boundary between them ($J^*(x^0) = 0$). Note that for all safe initial conditions vehicle A is guaranteed not to collide with vehicle B as long as it applies $u(t) = u^*(t)$ (starts decelerating) whenever the state reaches the boundary (i.e., whenever $J^*(x(t)) = 0$). For unsafe initial conditions, on the other hand, there exist actions of vehicles B and C where a collision between vehicles A and B is unavoidable, whatever vehicle A does.

The above calculations are used in the development of a computational spacing tool. The user of the tool is asked to provide the minimum deceleration rates, a_A^{min} and a_B^{min} , the minimum jerk j_A^{min} and the maximum allowable relative velocities of the collisions between vehicles B and D (v_B) and vehicles C and A (v_C)⁶. The tool then calculates the minimum spacing, x_3^0 , required to guarantee no collisions between vehicles A and B, for a given initial condition x_1^0, x_2^0 and x_4^0 , using Lemma 2 and analytical expressions for x_3^* and x_4^* derived from the dynamics of equation (1). The tool operation can be thought of as a function s :

$$\begin{aligned} s : \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (x_1^0, x_2^0, x_4^0) &\longmapsto x_3^0 = s(x_1^0, x_2^0, x_4^0) \end{aligned}$$

The function s is parameterized by the model parameters (braking capabilities, control loop lags, etc.). The maximum collision velocities v_B, v_C are used to set the initial relative velocity x_4^0 .

⁶For the more detailed models discussed in Appendix C the user needs to provide values for additional parameters such as the sensing and actuation delays and the jerk limits for vehicle B.

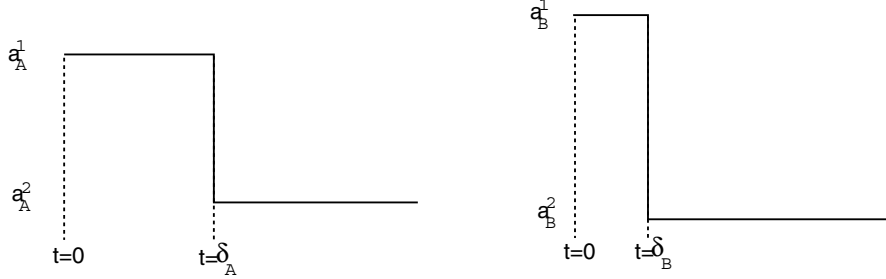


Figure 2: Assumed acceleration trajectories

Pure time delays incurred in sensing and actuation can be added at no extra cost by a simple modification of the tool input and output. Assume that the vehicles are originally in steady state ($x_2^0 = 0ms^{-2}$, $x_4^0 = 0ms^{-1}$) when vehicle B starts braking and that there is a delay of δ before vehicle A starts braking. Then the minimum spacing needed for safety is given by:

$$x_3^0 = s(x_1^0, 0, a_B^{min}\delta) - \frac{a_B^{min}\delta^2}{2} \quad (8)$$

(recall that $a_B^{min} < 0$). The results presented here assume that there are no first order actuation lags. These lags can be handled by the tool extensions presented in Appendix C.

3.3 Collision Tool

The spacing tool allows us to determine spacing requirements to guarantee that two vehicles will not collide even under emergency deceleration scenario. To be able to analyze the emergency deceleration of a platoon we also need to know what happens if these requirements are violated. Consider again the two vehicles A and B of Figure 1. To keep the calculations tractable we simplify the model for vehicle A slightly, by allowing its acceleration to change instantaneously ($j_A^{min} = -\infty$). Assume that the accelerations of A and B follow the trajectories of Figure 2 until the vehicles either stop or collide⁷. To prevent the vehicles from going backwards also assume that the acceleration of each vehicle becomes zero as soon as the vehicle stops. As $\dot{x}_B \geq 0$ and we are trying to investigate the cases where the vehicles collide, we restrict our attention to the interval of time when $\dot{x}_A > 0$. It is easy to show that under these conditions the spacing and relative velocity between vehicles A and B is given by:

$$x_3(t) = \frac{a}{2}t^2 + bt + c + x_3^0 \quad (9)$$

$$x_4(t) = at + b \quad (10)$$

The values of a, b , and c depend on the initial velocities of the vehicles and the parameters of the problem (a_i^j and δ_i with $i \in \{A, B\}$ and $j \in \{1, 2\}$). They are tabulated in Appendix A. If a collision takes place, equation (9) allows us to determine the time T at which it happens while equation (10) provides the relative velocity at impact.

To analyze cases where multiple collisions occur we would also like to determine the vehicle velocities after the collision. We model collision elasticity by a coefficient of restitution γ . If the collisions are centered, γ relates the longitudinal velocities before and after the collision by:

$$\gamma = \frac{\dot{x}_B(T^+) - \dot{x}_A(T^+)}{\dot{x}_A(T^-) - \dot{x}_B(T^-)} \quad (11)$$

⁷For spacing requirements for more elaborate deceleration profiles refer to [29].

The coefficient of restitution depends on the design of the vehicle body and bumpers. $\gamma = 1$ models perfectly elastic collisions whereas $\gamma = 0$ corresponds to plastic collisions. A second equation relating the speeds before and after collision is conservation of linear momentum. Let m_A and m_B denote the masses of the two vehicles and define $M = m_B/m_A$. Then:

$$\dot{x}_A(T^-) + M\dot{x}_B(T^-) = \dot{x}_A(T^+) + M\dot{x}_B(T^+) \quad (12)$$

Given values for γ and M equations (11) and (12) can be solved for $\dot{x}_A(T^+)$ and $\dot{x}_B(T^+)$ (which in turn gives $x_2(T^+)$ and $x_4(T^+)$) and the process can be repeated.

These calculations were implemented in a computational tool to analyze multiple collisions within a string of vehicles. The tool accepts as input the initial velocities, $x_{i_1}^0$, initial spacings $x_{i_3}^0$, acceleration levels a_i^j , $j = 1, 2$, delays δ_i , masses m_i and coefficients of restitution γ_i for each vehicle i in the string. It then solves equation (9) for all pairs of adjacent vehicles, determines the smallest collision time, T , and the vehicles involved, k and $k - 1$, calculates the state of all vehicles right before the collision, $x_i(T^-)$ for all i , solves equations (12) and (11) to obtain $x_k(T^+)$ and $x_{k-1}(T^+)$, and repeats the process. The iteration terminates when no more collisions are possible (the trailing vehicle stops). The tool returns the relative velocities of all the collisions that took place (and, upon request other statistics such as the relative velocity of the worst collision).

Note: The choice of trajectories for the accelerations of the two vehicles is motivated by physical considerations (such as actuator and communication delays) relating to the operation of the platoons. In addition this class of trajectories can be shown to contain trajectories which are in some sense optimal. A measure of the severity of the collision is the relative velocity at impact which can be encoded by the cost function:

$$J'(x^0, u, d) = |x_4(T)| \quad (13)$$

It can be shown [13] that the global saddle solution for the two player zero sum game over cost function J' involves u and \ddot{x}_B trajectories of the form shown in Figure 2.

3.4 Deceleration Capability Distributions

The output of both tools turns out to be very sensitive to the deceleration capabilities of the vehicles involved. To capture variations in deceleration capabilities among vehicles, we make use of two braking capability distributions. The first (Figure 3) was constructed by taking the average over manufacturer specifications for a number of current production models, each weighted by the sales figures for the corresponding model. An additional 30% degradation factor was added to account for temporal variations in deceleration capability due to tire and brake wear and changes in road conditions [11]. The resulting distribution leads to spacings that are conservative in terms of safety but somewhat pessimistic in terms of throughput. The second distribution (Figure 4) was constructed by averaging the braking capabilities obtained from consumer reports for a comprehensive selection of light duty passenger vehicles. As the tests in the reports were conducted with new cars on dry pavement, the results obtained using this distribution are somewhat optimistic in terms of throughput.

The two distributions are intended to capture lower and upper bounds on the system performance. We denote by $[\underline{a}, \bar{a}]$ the support of the distribution (i.e. the range of values for which the probability is nonzero). For numerical computations the support of each probability distribution is discretize into 11 “bins” of equal width. In our numerical experiments we typically assume the distribution of Figure 4 to be the nominal distribution.

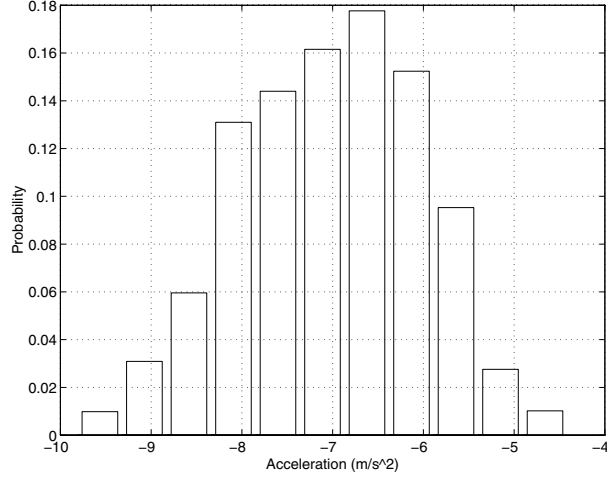


Figure 3: Pessimistic braking capability distribution

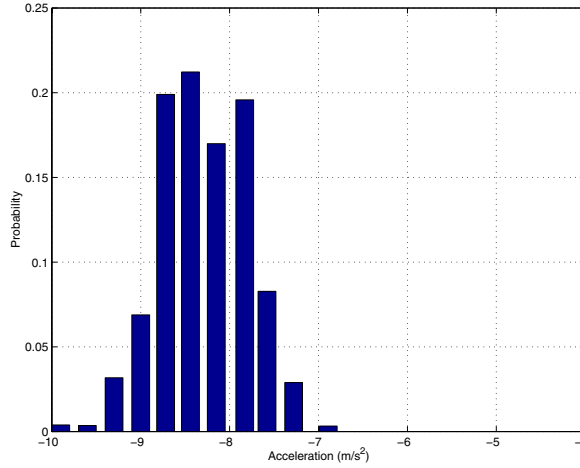


Figure 4: Optimistic braking capability distribution

4 Normal Mode Pipeline Throughput

4.1 Throughput Calculation

In the absence of faults and degraded environmental conditions we would like to be able to guarantee that there will be no collisions on the AHS. The spacing tool can be used to provide minimum spacing requirements in this case, which in turn can be translated to estimates of throughput. As no intra-platoon collisions occur during normal mode, $v_B = v_C = 0$ is used.

Assume that the traffic on the highway is arranged in platoons, all of them consisting of N vehicles and traveling at a velocity x_1^0 . If the average length of a vehicle is L , the average intra-platoon spacing is F and the average inter-platoon spacing is x_3^0 , then the pipeline capacity, Q of the highway in vehicles per lane per unit time is given by:

$$Q = \frac{N x_1^0}{x_3^0 + N L + (N - 1) F} \quad (14)$$

If we fix L and F , the spacing tool and equation (14) provide throughput estimates for a given

velocity x_1^0 and a platoon size N . The only additional information required is the relative deceleration capabilities, a_A^{min} and a_B^{min} , and the lower bound on the jerk j_A^{min} for each pair of platoons.

For platoons of two or more vehicles under normal operation, the deceleration exerted by the leader should also be limited by the deceleration capabilities of the followers, to guarantee the *string stability* of the platoon. Motivated by the work of [17] we use the following formula to calculate the minimum allowable deceleration, a^{allow} , for a leader:

$$a^{allow} = \max \left\{ a_0^{min}, \frac{a_1^{min}}{1.15}, \frac{a_2^{min}}{1.2}, \frac{a_3^{min}}{1.25}, \frac{a_i^{min}}{1.3} \text{ for } i \geq 4 \right\} \quad (15)$$

where a_i^{min} is the deceleration capability of the i^{th} follower (the leader if $i = 0$). For platoons of length $N < 5$ only the first N terms are considered.

4.2 Numerical Experiments

We use the distribution of Figure 4 as the nominal distribution for our numerical experiments. The values of 0.1 seconds and -75 m/s^3 are used for the delay δ and the minimum jerk j_{min}^A respectively. Assuming that the system starts at steady state ($x_2^0 = 0 \text{ ms}^{-2}$ and $x_4^0 = 0 \text{ ms}^{-1}$) the spacing tool is used to calculate the minimum value of x_3^0 needed to guarantee absence of inter-platoon collisions in each case. The resulting throughput values are then weighted according to the probability distribution and averaged.

One of the key parameters in our throughput study is the information structure assumed by the AHS. The crucial information in this case is deceleration capabilities. Each platoon may or may not have access to its own deceleration capability (a_A^{min}) or the deceleration capability of the platoon ahead (a_B^{min}). Four different cases can be distinguished. A platoon may:

1. have access to both pieces of information,
2. have access to its own deceleration capability but not that of the platoon ahead,
3. have access to the deceleration capability of the platoon ahead but not its own.
4. have access to neither,

The third alternative is probably unrealistic (at least in the absence of sensing faults) and will not be investigated further here. In order to guarantee safety for alternatives 2 and 4 a platoon has to assume the worst for the missing pieces of information. For a_A^{min} this implies assuming that our platoon can decelerate at the slowest possible rate. This amounts to setting $a_A^{min} = \bar{a}$ for single vehicles, $a_A^{min} = \bar{a}/1.05$ for two vehicle platoons, $a_A^{min} = \bar{a}/1.1$ for three vehicles platoons, etc. Similarly, to capture the worst case for the deceleration capability of the platoon ahead in terms of safety we need to assume $a_B^{min} = \underline{a}$. The results obtained for the three information structures for platoon sizes of $N = 1, 3, 5$ and 7 , $x_1^0 \in [0, 40] \text{ ms}^{-1}$ are shown in Figures 5–7.

As the spacing design is very sensitive to the relative braking capability of vehicles, the knowledge of braking capability allows the vehicles to follow very closely. However, it may not be possible to maintain a comfortable ride at such close following, as small errors in sensing and actuation would force the controller to apply maximum braking too frequently. To improve the ride quality a minimum time headway (h) requirement can be introduced. The spacing is then given by:

$$x_3^0 = \max\{s(x_1^0, 0, 0), hx_1^0\}$$

Figures 8, and 9 show the effect $h = 0.5s$ has on the throughput.

4.3 Discussion

The figures indicate that if full information about the vehicles deceleration capabilities is not available much higher throughput is possible using large platoons. The numbers should be compared to the current throughput for manual traffic, which peaks at about 1800–2000 vehicles per lane per hour. One should be careful, however, as the manual throughput value includes lateral flow disturbances that were neglected here. If deceleration capability information is available however, the throughput obtained for all platoon sizes up to $N = 7$ is comparable. The throughput is reduced if minimum headway requirements are considered, especially for smaller platoon sizes.

It should be noted that in the case where both deceleration capabilities are known, our experiments exhibit a bias in favor of smaller platoons. For simplicity, the probability distribution for the deceleration exerted by the last vehicle of the platoon was assumed to be the same as the deceleration capability distribution for a single vehicle. In fact this distribution should exhibit a bias towards milder decelerations for platoons of size $N > 1$. The reason is that as the platoon gets larger, the probability that the last vehicle is not the worst (least capable of decelerating) becomes larger. However, there is no straightforward way of determining exactly how much this last vehicle will have to decelerate as a function of the capabilities of the vehicles ahead; the value depends on the actual controller design. The scaling factors of equation (15) could be used as an indication, this was not done however, primarily because of limits in computational power. Therefore the results in the cases when both deceleration capabilities are known should be considered more for their qualitative than their quantitative value.

A number of interesting facts can be inferred from the figures. For example, in Figure 6 the incremental improvement with respect to platoon size decreases as the platoon size increases. This is due to the fact that the maximum deceleration applied by the leader of a platoon under the string stability requirement of (15) is essentially determined by the deceleration capability of the worst vehicle in the platoon. The probability of a really bad vehicle within a platoon increases as the platoon size increases. This effect is not observed in Figure 7, because the worst case deceleration capability assumed there levels off after the fourth follower. This is also the reason why the improvement from $N = 5$ to $N = 7$ is greater than the one from $N = 3$ to $N = 5$ in Figure 7.

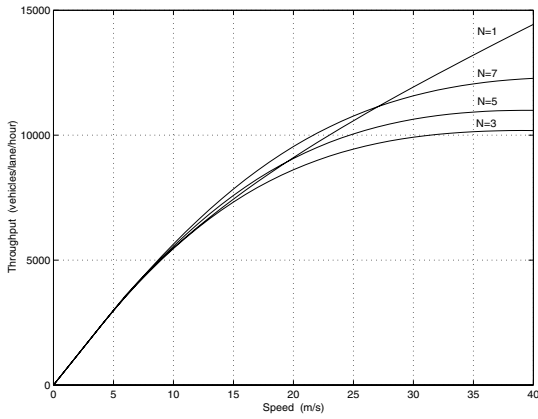


Figure 5: Throughput when both deceleration capabilities are available

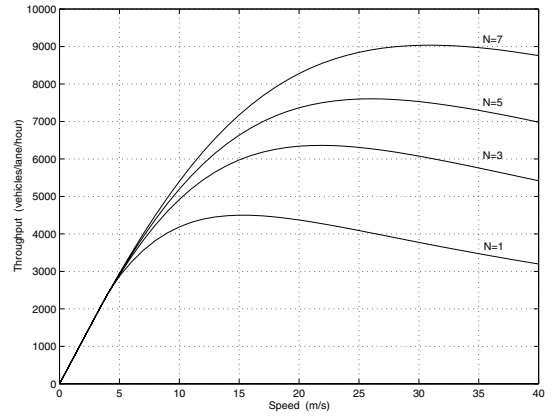


Figure 7: Throughput when no deceleration capability is available

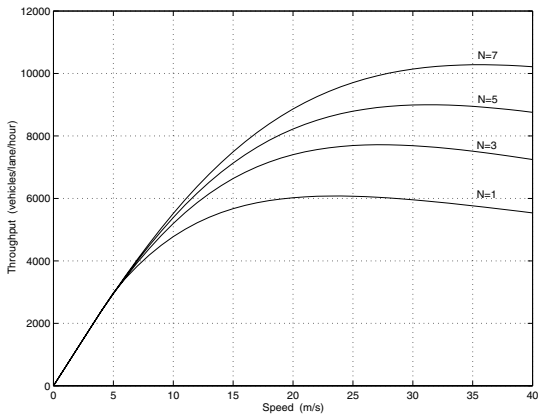


Figure 6: Throughput when own deceleration capability is available

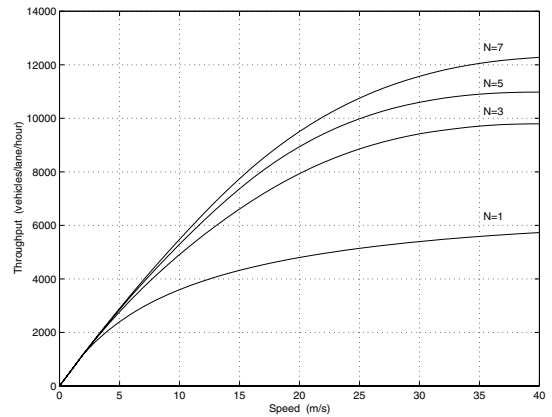


Figure 8: Throughput with minimum headway of 0.5 seconds when both deceleration capabilities are available

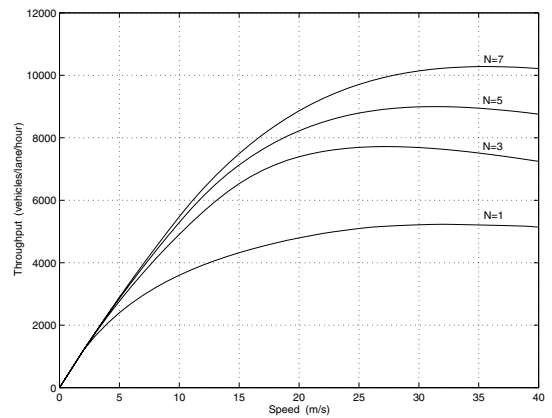


Figure 9: Throughput with minimum headway of 0.5 seconds when own deceleration capability is available

5 Collisions Due to Emergency Braking

5.1 Emergency Braking

Next we investigate the intra-platoon collisions that may occur during an incident of emergency deceleration of a platoon of vehicles. As an emergency braking system for platoons has not been designed yet, a simple control scheme is considered: follower i ($i = 0$ for the leader) keeps a constant acceleration x_{i2}^0 until a time δ_i when it switches to its minimum deceleration a_i^{min} . The time δ_i may depend on the processing and actuation delays, δ_a , as well as the communication architecture within a platoon. For hop-by-hop communication a delay of δ is added for each follower (i.e., $\delta_i = i \cdot \delta + \delta_a$). For broadcast communication, on the other hand, the delay is δ for all the followers (i.e., $\delta_0 = \delta_a$, $\delta_i = \delta + \delta_a, i > 0$). For simplicity we only consider the case $\delta_a = 0$ in the numerical experiments.

5.2 Numerical Experiments

The collision tool can be used to obtain collision statistics for platoons of various sizes. The subsequent figures reflect expected values of the quantities over the probability distributions of Figure 3 and 4. Unless otherwise stated it will be assumed that the platoon is initially at steady state with $x_{i1}^0 = 25ms^{-1}$, $x_{i2}^0 = 0ms^{-2}$, $x_{i3}^0 = 1m$ and $x_{i4}^0 = 0ms^{-1}$ for all i , that all vehicles have equal mass $m_i = 1500Kg$ and that all collisions are elastic ($\gamma = 1$). The default communication architecture will be hop-by-hop with a delay of $\delta = 0.05s$. The collisions will be classified according to their relative velocity at impact, which is a measure of their severity [23].

Figure 10 shows the percentage of collisions that on the average fall into classes (starting at $0ms^{-1}$ and each class having a width of $0.3ms^{-1}$). The figure indicates that even though most of

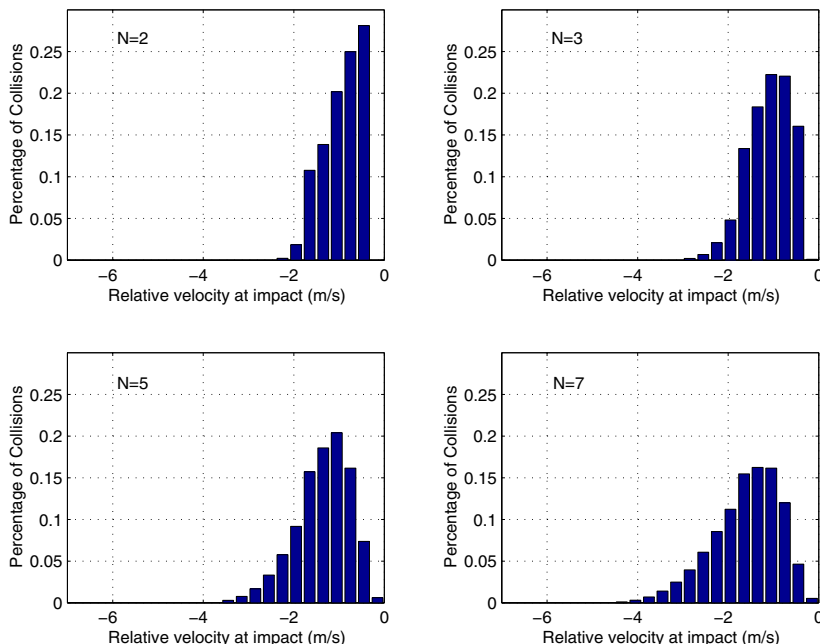


Figure 10: Classification of collisions by relative velocity

the collisions occur at small relative velocities, there is still a significant probability of high relative

velocity collisions⁸. This probability increases with the size of the platoon. Some more statistics extracted in the experiments are given in Figure 11. Note that the average number of collisions per vehicle due to hard braking by the platoon leader increases roughly linearly with the platoon size.

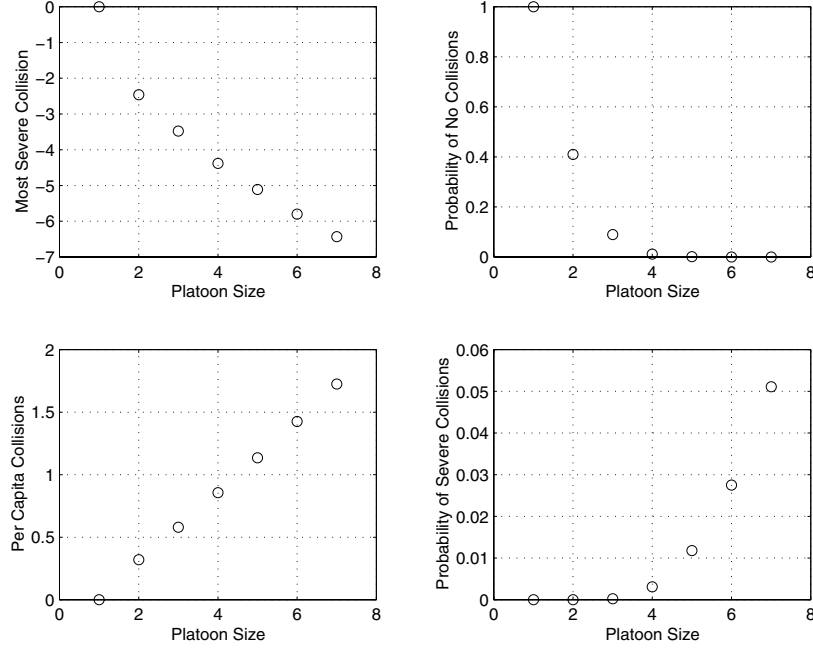


Figure 11: Other Collision Statistics

More experiments were conducted to investigate the sensitivity of the collision statistics to various design parameters. The corresponding figures are presented in Appendix B. First we considered sensitivity to the communication architecture and the speed of operation. Figure 17 shows collision statistics for 5 car platoons with variations in communication type, communication delay and nominal speed. All figures look more or less similar to the nominal case, indicating that the sensitivity of the statistics to these parameters is low.

Then sensitivity to follower spacing was considered. The results for five car platoons with variation in intra-platoon spacing (IP) are shown in Figure 18. Clearly the statistics are much more sensitive to this parameter.

To make the model more realistic, a coefficient of restitution that depends on the relative velocity of impact (δv) was introduced. Motivated by the work of [30] we use the following formula to calculate the coefficient of restitution for a collision with relative velocity δv :

$$\gamma = \begin{cases} 1 + \frac{0.9 \delta v}{|v_\gamma|} & \text{if } v_\gamma \leq \delta v \leq 0 \text{ms}^{-1} \\ 0.1 & \text{if } \delta v \leq v_\gamma \end{cases} \quad (16)$$

The results for 5 car platoons with variations in v_γ are shown in Figure 19. The statistics seem to be sensitive to this parameter as well.

Finally, to investigate sensitivity to changes in the deceleration distribution, we use the larger width braking distribution of Figure 3. The results, summarized in Figures 20 and 21, indicate that the statistics are also sensitive to this parameter.

⁸The value of 3ms^{-1} is quoted in [23] as a limit for acceptable relative velocity at impact.

Some further experiments were conducted to establish trends for the parameters to which the statistics seem to be the most sensitive. The results are summarized in Table 1. Plots illustrating these trends are given in Appendix B.

Parameter Variation	No collision Probability	per capita Collisions	worst case Collision speed	$P(\Delta v < -3ms^{-1})$
Vary intra-platoon spacing				
1m	0.001	1.13	-5.11	0.0118
2m	0.033	0.776	-7.045	0.113
3m	0.186	0.532	-8.55	0.2054
5m	0.481	0.2678	-7.65	0.5102
Vary coeff. of rest. v_γ				
elastic	0.001	1.13	-5.11	0.0118
-6.5 m/s	0.001	1.37	-3.9	0.0011
-4.5 m/s	0.001	1.72	-3.6	0.0004
Vary braking distribution				
Nominal	0.001	1.13	-5.11	0.0118
Broad width	0.0016	1.37	-6.36	0.0684
IP=2m, vary v_γ				
elastic	0.033	0.776	-7.045	0.113
-6.5 m/s	0.033	0.8705	-5.1	0.043
-4.5 m/s	0.033	1.135	-4.8	0.0269
$v_\gamma = -6.5m/s$, vary IP				
1m	0.001	1.37	-3.9	0.0011
2m	0.033	0.8705	-5.1	0.043
3m	0.1864	0.5815	-6.3	0.1492
5m	0.4812	0.2885	-6.6	0.5

Table 1: Collision statistics

5.3 Discussion

The numerical experiments indicate that even though collisions take place primarily at low relative velocities, there is still a substantial number of severe collisions. Moreover *the number of collisions and their severity increases as the platoon size increases*. The collision statistics are rather insensitive to the communication architecture and the speed of operation. A slight improvement is obtained if the broadcast architecture is used and the communication delays are reduced and if the speed of operation is lower. However the changes are rather small, at least for realistic parameter ranges.

The statistics are very sensitive to changes in the intra-platoon separation, the elasticity of the collisions and the deceleration distribution. An increase in intra-platoon separation decreases the frequency of the collisions but increases their severity. A decrease in elasticity on the other hand seems to decrease both collision frequency and severity. The same is true about a decrease in the width of the deceleration distribution. It should be noted that these trends reflect changes *on the average*. Particular runs may exhibit very different behavior; they may be sensitive to the “insensitive” parameters, exhibit reductions where increases are observed on the average, etc.

The results in this section can be especially useful in guiding technology and policy decisions. For example, our results indicate that eliminating vehicles with low braking capability from the AHS (e.g., by using tighter check-in procedures) should on the average result in an improvement in the severity and number of collisions observed in emergency situations. A similar improvement is expected if the collisions are made less elastic (by appropriate bumper design for example) and if the vehicles in a platoon are packed more closely (the number of collisions will increase in this case, but their severity will decrease). Similar use can be made of the trends observed in the remaining parameters.

Finally a technical note. The collision statistics need to be averaged over a relatively large number of collisions to “settle down”. In cases where a small number of collisions is observed (for example if the platoons are small or the intra-platoon spacing is large) the deceleration distribution has to be sampled more densely for the statistics to be accurate. In cases where many collisions are observed the sampling may be coarser.

6 Relating Collisions and Throughput

6.1 Throughput in the presence of collisions

The collision calculation imposes a further restriction on the throughput that can be achieved safely. To ensure that the collisions that occur because of emergency braking do not propagate beyond the platoon that executes the maneuver, additional inter-platoon spacing is needed. The amount of extra spacing necessary can be calculated by obtaining the relative velocities of the most severe collision experienced by the leader and the last vehicle in the platoon (using the collision tool). These relative velocities can then be used as v_C and v_B respectively in the spacing tool calculations. The result will be an increase in the inter-platoon spacing to guarantee that collisions of one platoon do not affect another, and a consequent reduction in throughput.

Note that the disturbance to platoon A generated by hard braking of platoon B along with collisions within platoons A and B, although rare is not unrealistic. The leader of platoon B may be required to brake hard in response to obstacles or failures. This uncoordinated hard braking by the leader of platoon B may give rise to intra-platoon collisions as seen in the previous section. At the same time, platoon C may be accelerating to join platoon A, in which case a collision between A and C can not be ruled out [13]. Doing the spacing calculations assuming the worst-case disturbance guarantees that collisions do not propagate upstream even in this extreme but hopefully rare situation (as shown in [13]). Note that, the frequency of obstacles and failures will depend on vehicle and system reliability whereas the frequency of joins and splits will depend on the entry/exit patterns and demand level.

6.2 Numerical Experiments

The throughput obtained assuming the default platoon setup of Section 5 are summarized in Figure 12. Figures 13 and 14 provide the data when information about deceleration capability is available. Figures 15 and 16 show data for the above cases with an additional minimum headway requirement of 0.5s. As expected the throughput is severely reduced in all cases except individual vehicles (compare Figures 5-9 with Figures 12-16). In fact, because the severity of collision increases as the platoon size increases, the throughput obtained for all platoon sizes up to $N = 5$ are comparable.

The results on throughput under collisions depend on both the parameters that affect the pipeline calculations and the parameters that affect the collision calculations. Therefore sensitivity analysis becomes a complicated endeavor. To simplify it somewhat we first observe that

we can effectively ignore the dependence of the collision statistics on the speed of operation. The collision distribution along with the worst-case collision severity does not change over a wide range of speeds of interest. The numerical experiments show that this result holds for all combinations of intra-platoon separation, braking distribution and collision elasticity. Therefore, using the collision statistics obtained at 25ms^{-1} operating speed results is a reasonable approximation for operating speeds in the range $15 - 40\text{ms}^{-1}$.

First we consider variations in the three most sensitive parameters for the collision statistics, namely intra-platoon spacing, braking distribution and elasticity of collisions. Figures 25-27 (Appendix B) show the variation of throughput with intra-platoon separation (IP) for perfectly elastic collisions, nominal braking distribution and $N=5$. *As intra-platoon separation increases the pipeline throughput decreases because of increased intra-platoon spacing and increased severity of collisions resulting in increased inter-platoon spacing.*

Next, we consider variation in braking capability with 1m intra-platoon spacing and elastic collisions. Figures 28-32 show the effect of change in braking distribution (braking distribution of Figure 3) on the pipeline capacity. Comparing these plots with Figures 12-16, we observe that *both the individual vehicles and platoons benefit from the reduced width of the braking distribution; with individual vehicle AHS still providing comparable throughput to 5 car platoons.*

Finally, we look at variation of throughput with collision elasticity at 1m intra-platoon separation, $N=5$ and nominal braking distribution. We consider three cases; elastic collisions, $v_\gamma = -6.5\text{m/s}$, and $v_\gamma = -4.5\text{m/s}$. Figures 33-35 show the outcome of this experiment. We see that *as the collisions become more and more inelastic, the throughput increases as the severity of the worst case collisions decreases.*

Until now, we assumed that the actual braking capability can be estimated on-line within an error bound of one bin-width (recall that the probability distribution is divided into 11 bins). Given the sensitivity of throughput to this parameter, we investigate the effect of errors in estimating the braking capability. Figures 36-41 illustrate the effects of capability estimation errors on throughput. We observe that *the braking capability estimation error results in reduction of throughput for small sized platoons but does not affect the throughput of large size platoons.*

6.3 Discussion of Results

At moderate capacities of approximately 5000 vehicles per lane per hour, an individual vehicle AHS with knowledge of (at least its own) braking capability can be designed to guarantee no collisions (for the emergency deceleration scenarios considered here at least). The throughput obtained in this case is comparable to the throughput for 5 vehicle platoons (even with the larger width braking distribution); however, intra-platoon collisions during emergency deceleration can not be ruled out. Throughput can be improved by reducing the intra-platoon following distance, reshaping the braking distribution (not allowing vehicles with low braking capability to enter the AHS) and by making bumpers less elastic. A narrower braking distribution benefits both platoons and individual vehicles by reducing the average following distance. Bumper design and intra-platoon following of course have no effect for individual vehicles. Note that reliably operating platoons at spacings as low as 1m may prove to be a major technological challenge.

If braking capability can not be estimated, both platooning and individual vehicle throughput reduce substantially. In fact, inter-platoon separations of up to 150m may be required to guarantee safety (for the larger width braking distribution). This implies the need for expensive forward looking range and range rate sensors. To achieve higher capacity, there is no other choice but to implement large size platoons. In that case, severe intra-platoon collisions can not be ruled out (though their probability is low). It is also not clear under what situation (urban, inter-urban, rural)

can a large average platoon size be maintained. Recall that a platoon based AHS would incur further reductions from the pipeline throughput due to joining and splitting of platoons.

The throughput calculated in this way will probably be unnecessarily pessimistic. The spacings produced by the above calculation will guarantee that collisions do not propagate from one platoon to the next in any situation (even if both platoons have to undergo emergency braking and the deceleration capabilities of the followers are arranged in the the worst possible order). Faults that require emergency braking are likely to be rare and so is the worst deceleration ordering. It may therefore be excessive to continuously demand such a severe throughput sacrifice to ensure that nothing will go wrong in the case of a very rare event. A more thorough investigation of the tradeoffs between safety and throughput will be needed in this case. In addition, the collision statistics were collected for a natural but definitely not optimal emergency deceleration strategy. Improvement, both in the number and the severity of collisions, may be possible with a better design.

The calculations in this section exhibit small biases for a number of reasons. The throughput calculations assume that the first and last vehicle experiences exactly one collision per incident when in fact they may experience multiple collisions. Also, all collision statistics assumed that the emergency deceleration originates with the leader of the platoon. Though unlikely, it is possible that even worse collisions may be obtained in other cases, if one of the followers develops a fault for example. In the case of multiple inelastic collisions, the bumpers are likely to get damaged in the first collision. Therefore the elasticity of the next collision will be affected. Finally, making bumpers less elastic will also lead to more repair cost per incident.

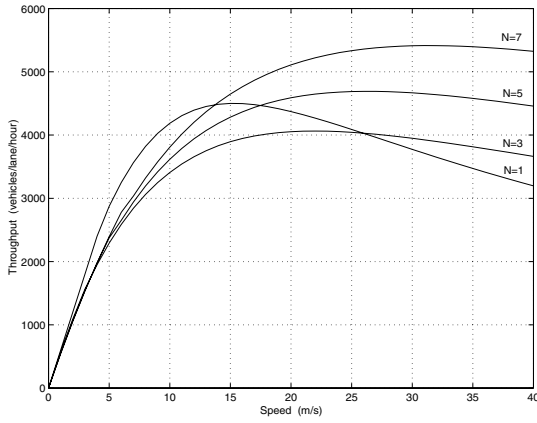


Figure 12: Reduction in throughput because of collisions

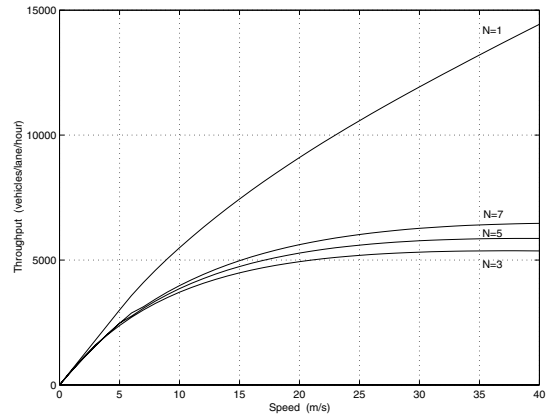


Figure 14: Reduction in throughput because of collisions: both capabilities known

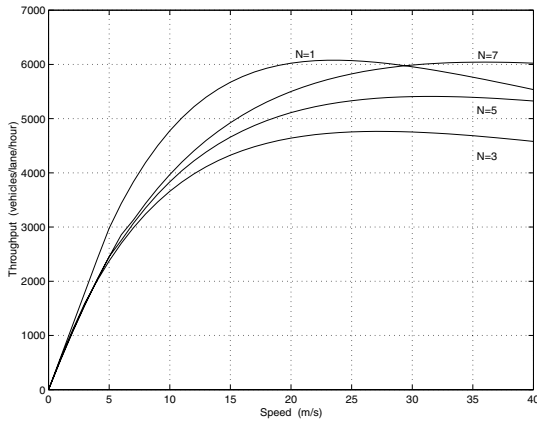


Figure 13: Reduction in throughput because of collisions:own capability known

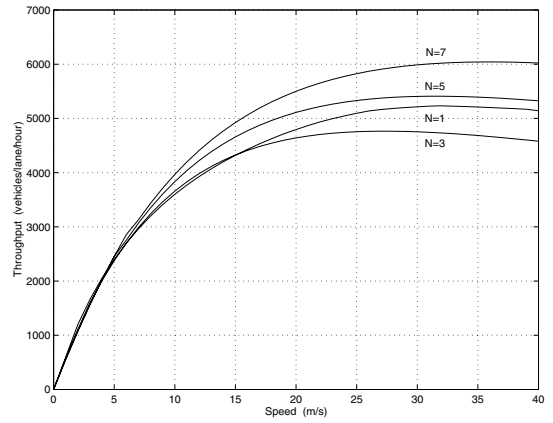


Figure 15: Reduction in throughput because of collisions:own capability known, minimum headway 0.5s

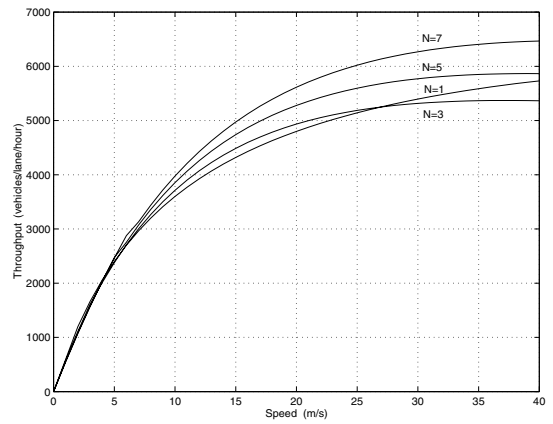


Figure 16: Reduction in throughput because of collisions: both capabilities known, minimum headway 0.5s

7 Concluding Remarks

We presented a methodology for performing safety and throughput analysis for AHS concepts. Our methodology is based on analytical calculations, implemented on software tools. We showed how these tools can be used to compare the performance of various proposed AHS architectures, making use of minimal assumptions about the distribution of intelligence among the vehicles and their dynamical models.

Our analysis indicates that an AHS consisting of dedicated lanes carrying individual vehicles with knowledge of their own braking capability can lead to a pipeline capacity of the order of 5000 vehicles per lane per hour. Compared to today's manually driven highways, such an AHS can carry twice the number of vehicles at highway speed with a substantial improvement in safety. A further increase in capacity requires formation of multi-vehicle platoons. To obtain a throughput comparable to the one for individual vehicles an AHS populated with 5 car platoons is needed. Formation of platoons requires more inter-vehicle communication to guarantee string stability within the platoon and to execute maneuvers (joining and splitting of platoons and lane changing). An AHS based on individual vehicles has the additional advantage of not producing any inter-vehicle collisions in case of a hard deceleration emergency; intra-platoon collisions can not be ruled out in cases of hard braking. Finally, platoon formation and dissipation introduces additional losses in capacity that are not present with individual vehicles. The loss of capacity due to lane changing is also likely to be higher. Capacity and safety under platooning can be improved by forming platoons only with vehicles of similar braking capabilities, by using very tight intra-platoon spacing, accurate braking capability estimation and inelastic bumpers.

The results of this study can be used to deduce design and technology requirements. They can also be used to assist policy decisions such as a choice between individual vehicle and platooned AHS, or obtaining selective entrance policies that would yield high throughput and be socially fair.

The main limitation of the work presented here is that it ignores the lateral motion of the vehicles, both as a source of throughput degradation and as a source of incidents (lane departures). The effect of lateral flow on throughput has been studied somewhat in the literature [6, 7, 8, 9, 10]. Lateral motion as a source of incidents is harder to study in the framework considered here. Unlike hard deceleration incidents, the effect of lane departures can not be eliminated by introducing additional spacing. Still, the introduction of additional spacing has a positive effect as it decreases the probability that a lane departure will cause an accident in the target lane. In this context, a platoon based AHS has an advantage over an AHS based on individual vehicles, as the inter-platoon gaps will be on the average much larger than the inter-individual vehicle gaps for the same level of throughput. It is unclear at this stage whether this advantage outweighs the fact that platoons are susceptible to collisions due to hard deceleration and the fact that the result is likely to be much more catastrophic if a lane departure causes an accident with a platoon. In our future work we intend to address this issue by estimating the probability of the lane departure and hard braking incidents, fixing the flow at a given level less than the pipeline throughput calculated here and developing queuing model to account for the extra space. We will then use the computational tools presented here to estimate the number and severity of collisions due to both classes of incidents for different AHS concepts. Because the number of probabilistic parameters is much larger in this case we may have to resort to Monte Carlo style runs of the tools to collect statistics.

We envision our methodology as a first step towards abstracting the macroscopic, emergent behavior of the automated highway system from the microscopic, local interactions between the vehicles. The tools rely on probabilistic models for the vehicle parameters and can therefore be thought of as a first step towards the probabilistic verification of the hybrid control architectures for automated highways, both under normal conditions and in the presence of faults. All these problems

should be of interest not just for automated highways, but also for other large scale systems such as air traffic management systems, power networks, etc.

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A Spacing Equation

As discussed in Section 3 the spacing between two vehicles following the deceleration profiles of Figure 2 is given by the equation:

$$x_3(t) = \frac{a}{2}t^2 + bt + c + x_3^0$$

Let \dot{x}_A and \dot{x}_B denote the initial velocities of vehicles A and B and define:

$$T_{A1} = \begin{cases} -1 & \text{if } a_A^1 \geq 0 \\ -\frac{\dot{x}_A}{a_A^1} & \text{if } a_A^1 < 0 \end{cases} \quad T_{A2} = \begin{cases} -1 & \text{if } a_A^2 \geq 0 \\ \delta_A - \frac{a_A^1 \delta_A + \dot{x}_A}{a_A^2} & \text{if } a_A^2 < 0 \end{cases} \quad (17)$$

$$T_{B1} = \begin{cases} -1 & \text{if } a_B^1 \geq 0 \\ -\frac{\dot{x}_B}{a_B^1} & \text{if } a_B^1 < 0 \end{cases} \quad T_{B2} = \begin{cases} -1 & \text{if } a_B^2 \geq 0 \\ \delta_B - \frac{a_B^1 \delta_B + \dot{x}_B}{a_B^2} & \text{if } a_B^2 < 0 \end{cases} \quad (18)$$

The values of a , b and c depend on the deceleration trajectory parameters and the time t . They are summarized in the following table:

a	b	c	t
$a_B^1 - a_A^1$	$\dot{x}_B - \dot{x}_A$	0	$C_1 \wedge C_3$
$-a_A^1$	$-\dot{x}_A$	$\frac{a_B^1 T_{B1}^2}{2} + \dot{x}_B T_{B1}$	$C_1 \wedge C_4$
$a_B^2 - a_A^1$	$(a_B^1 - a_B^2)\delta_B + \dot{x}_B - \dot{x}_A$	$\frac{(a_B^2 - a_B^1)\delta_B^2}{2}$	$C_1 \wedge C_5$
$-a_A^1$	$-\dot{x}_A$	$-\frac{(a_B^1 \delta_B + \dot{x}_B)^2}{2a_B^2} + \frac{a_B^1 \delta_B^2}{2} + \dot{x}_B \delta_B$	$C_1 \wedge C_6$
$a_B^1 - a_A^2$	$(a_A^2 - a_A^1)\delta_A + \dot{x}_B - \dot{x}_A$	$-\frac{(a_A^2 - a_A^1)\delta_A^2}{2}$	$C_2 \wedge C_3$
$-a_A^2$	$(a_A^2 - a_A^1)\delta_A - \dot{x}_A$	$\frac{a_B^1 T_{B1}^2}{2} + \dot{x}_B T_{B1} - \frac{(a_A^2 - a_A^1)\delta_A^2}{2}$	$C_2 \wedge C_4$
$a_B^2 - a_A^2$	$(a_B^1 - a_B^2)\delta_B + (a_A^2 - a_A^1)\delta_A + \dot{x}_B - \dot{x}_A$	$\frac{(a_B^2 - a_B^1)\delta_B^2}{2} - \frac{(a_A^2 - a_A^1)\delta_A^2}{2}$	$C_2 \wedge C_5$
$-a_A^2$	$(a_A^2 - a_A^1)\delta_A - \dot{x}_A$	$-\frac{(a_B^1 \delta_B + \dot{x}_B)^2}{2a_B^2} + \frac{a_B^1 \delta_B^2}{2} + \dot{x}_B \delta_B - \frac{(a_A^2 - a_A^1)\delta_A^2}{2}$	$C_2 \wedge C_6$

Table 2: Table of Coefficients

Let \wedge and \vee denote logical “AND” and “OR” respectively. $C_i, i = 1, \dots, 6$ are predicates on the time, t , defined by:

$$\begin{aligned}
 C_1 &= t \leq \delta_A \wedge (t \leq T_{A1} \vee T_{A1} < 0) \\
 C_2 &= t > \delta_A \wedge t \leq T_{A2} \wedge T_{A2} \geq 0 \wedge (\delta_A \leq T_{A1} \vee T_{A1} < 0) \\
 C_3 &= t \leq \delta_B \wedge (t \leq T_{B1} \vee T_{B1} < 0) \\
 C_4 &= t > T_{B1} \wedge \delta_B > T_{B1} \wedge T_{B1} \geq 0 \\
 C_5 &= t > \delta_B \wedge t \leq T_{B2} \wedge T_{B2} \geq 0 \wedge (\delta_B \leq T_{B1} \vee T_{B1} < 0) \\
 C_6 &= t > T_{B2} \wedge (\delta_B \leq T_{B1} \vee T_{B1} < 0)
 \end{aligned}$$

B Sensitivity Analysis Plots

B.1 Collisions Due to Emergency Braking

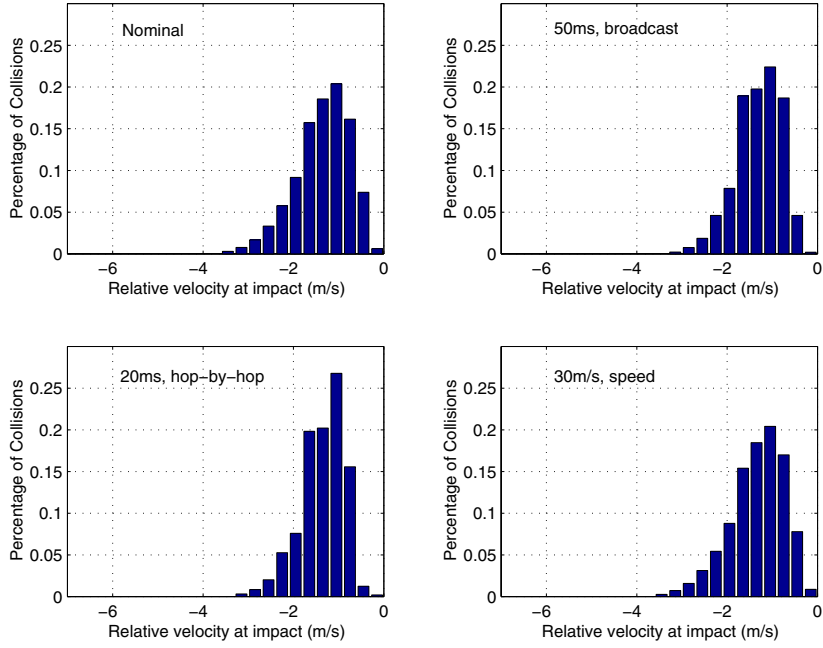


Figure 17: Variations in communication type, delay and nominal speed, $N = 5$

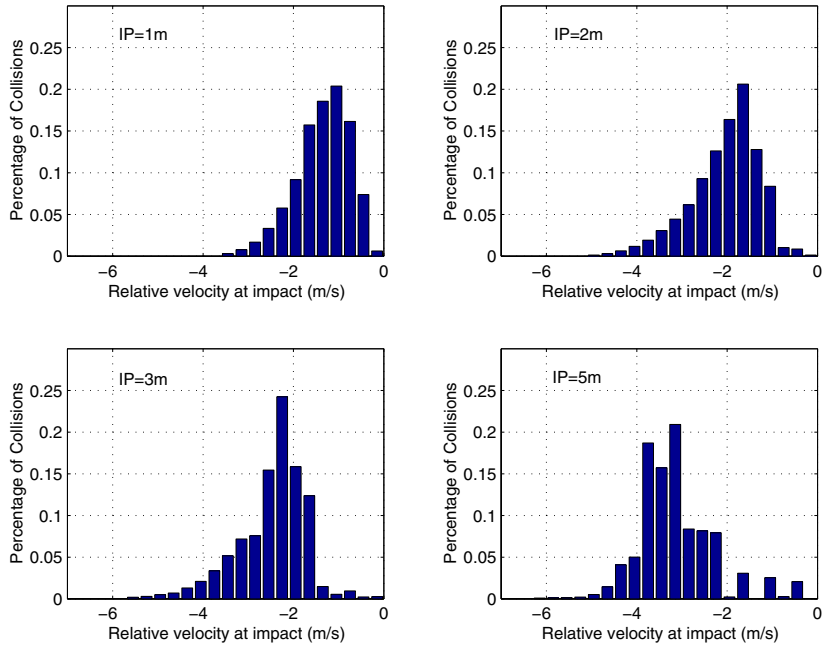


Figure 18: 5 car platoons with variation in intra-platoon spacing (IP)

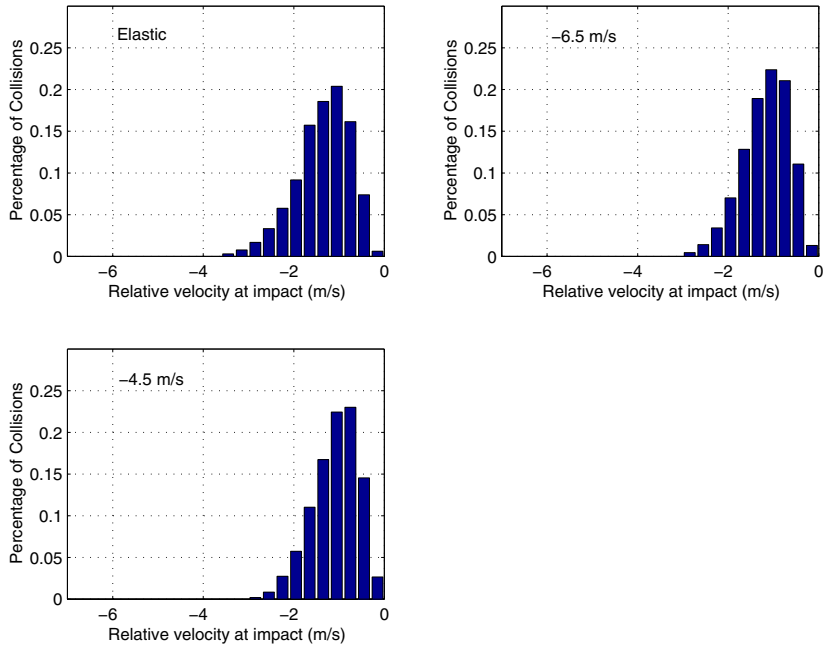


Figure 19: Variation in collision elasticity parameter v_γ

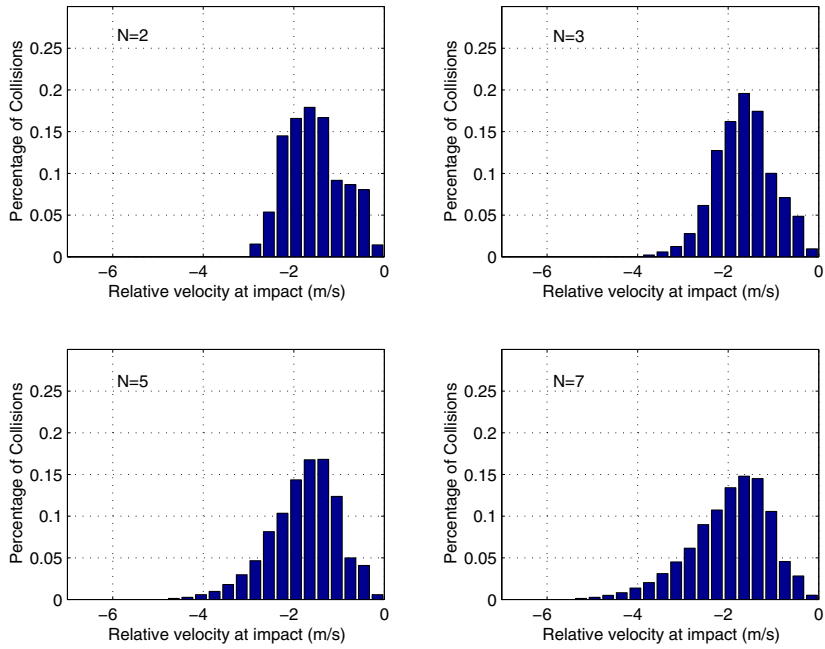


Figure 20: Variation in braking distribution (Distribution of Figure 3)

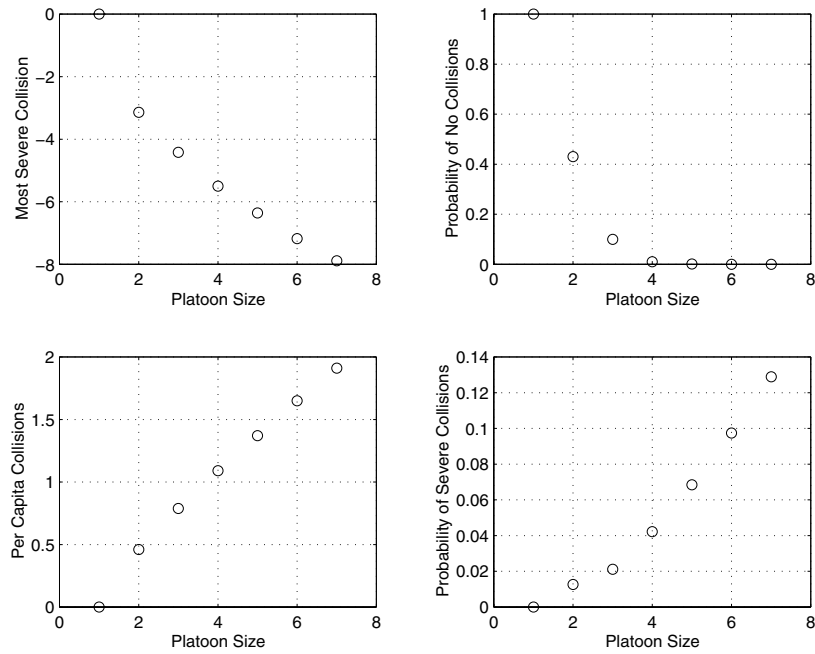


Figure 21: Collision statistics with distribution of Figure 3

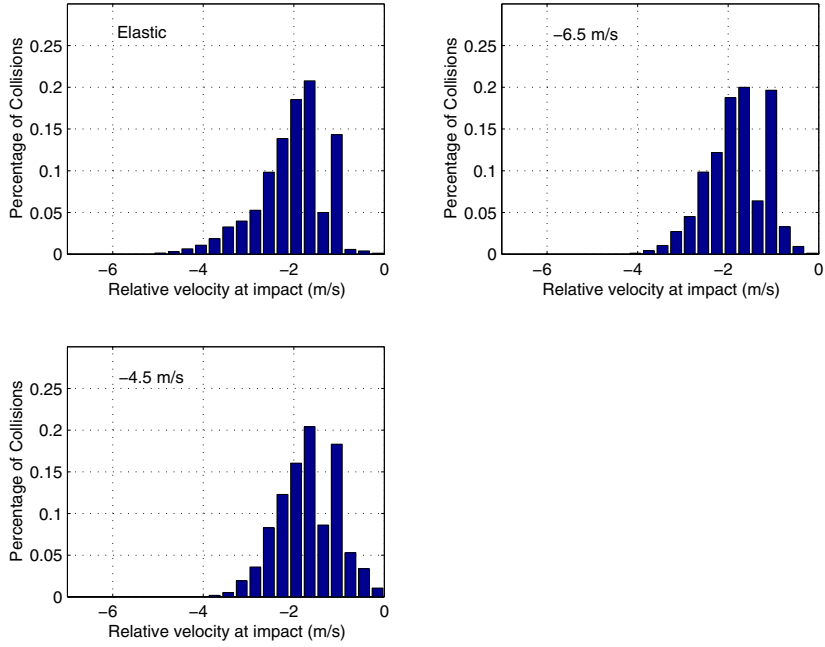


Figure 22: Variation in v_γ with IP=2m, N=5

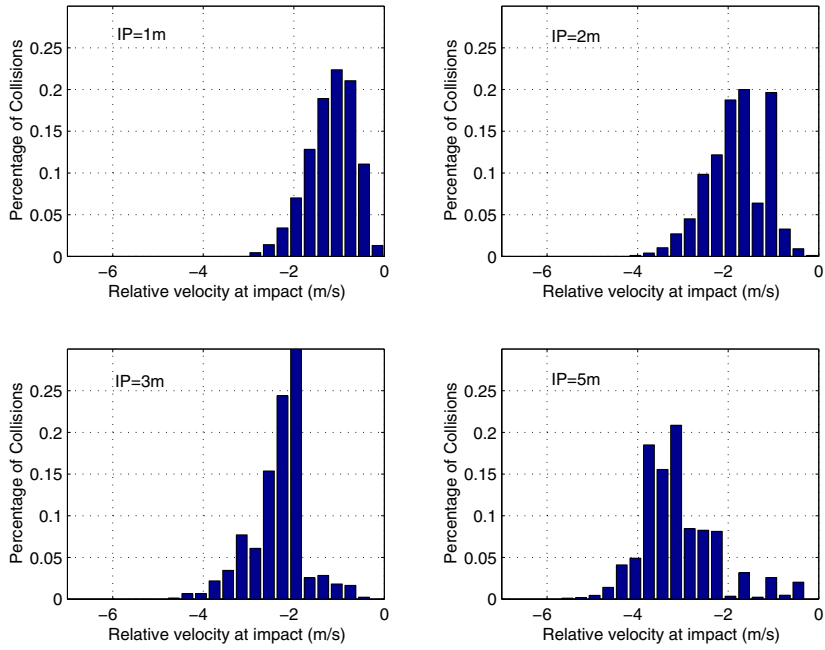


Figure 23: Variation in IP with $v_\gamma = -6.5\text{m/s}$, N=5

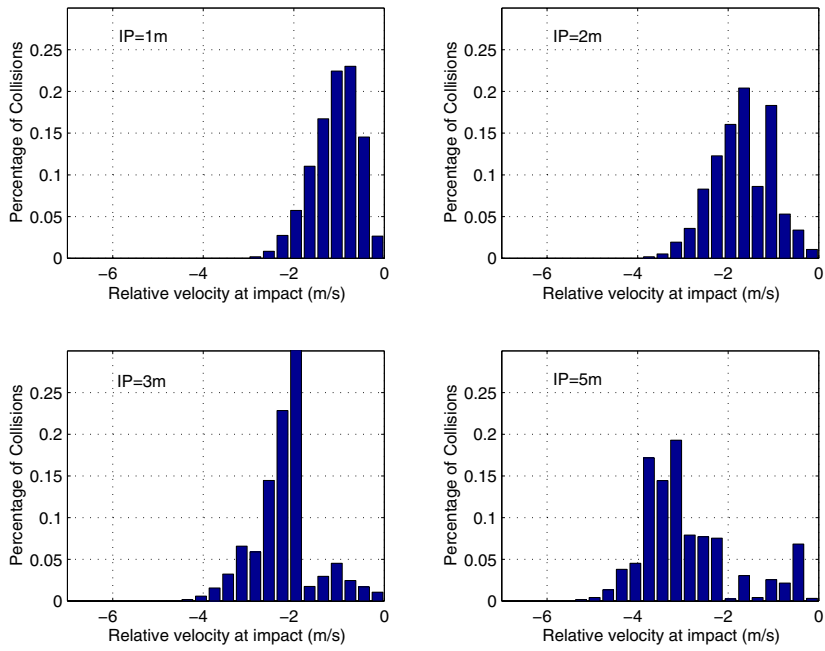


Figure 24: Variation in IP with $v_\gamma = -4.5m/s$, $N=5$

B.2 Throughput under Collisions

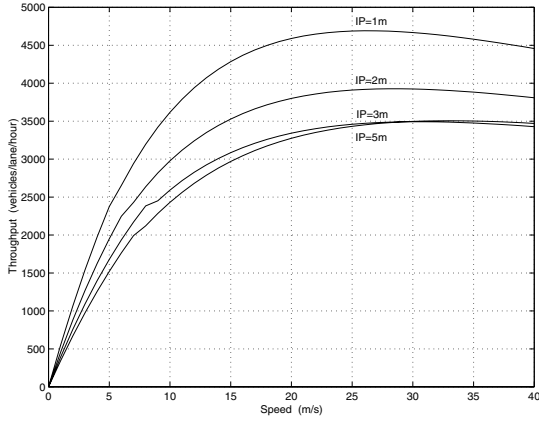


Figure 25: Variation of throughput with intra-platoon separation (IP)

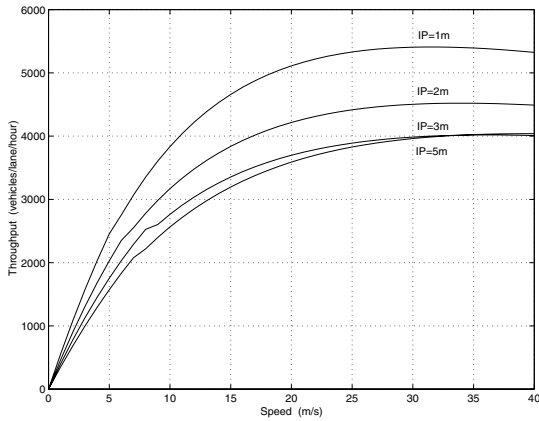


Figure 26: Variation of throughput with intra-platoon separation (IP): own braking capability known, minimum headway 0.5s

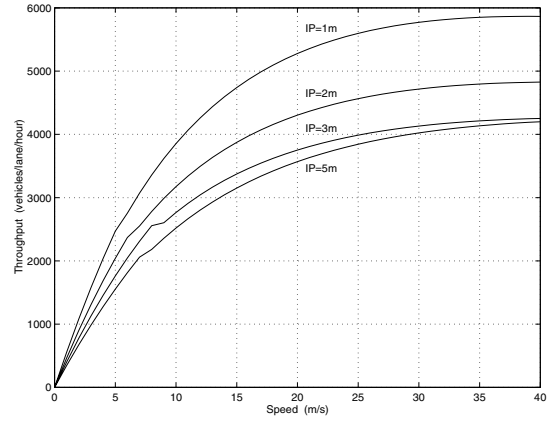


Figure 27: Variation of throughput with intra-platoon separation (IP): both capabilities known, minimum headway 0.5s

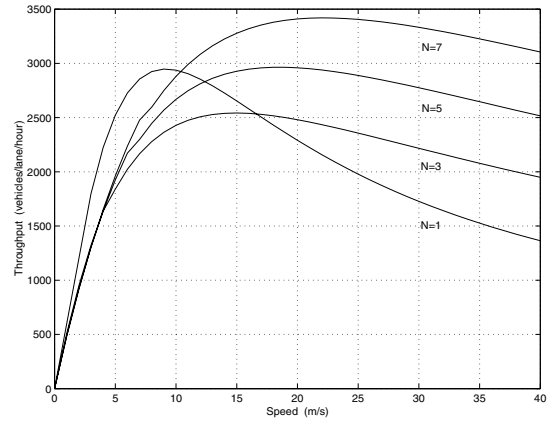


Figure 28: Throughput with large width braking distribution

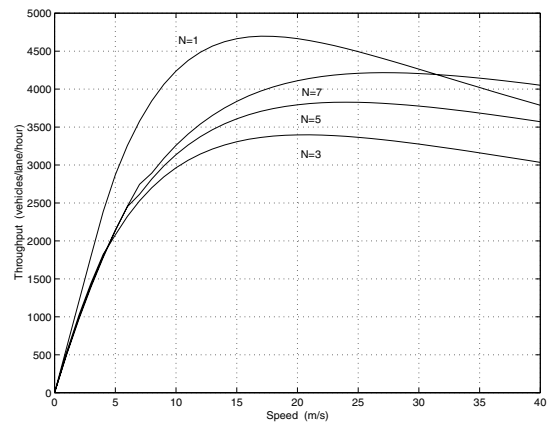


Figure 29: Throughput with large width braking distribution: own capability known

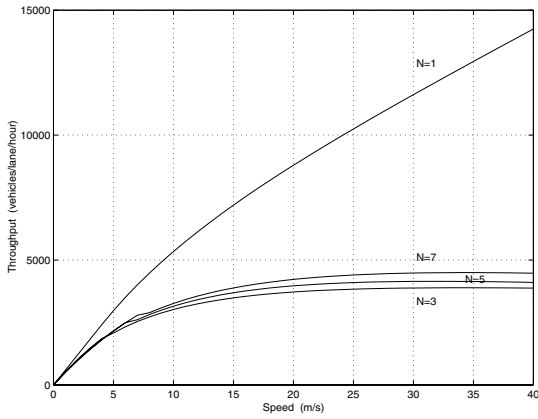


Figure 30: Throughput with large width braking distribution: both capabilities known

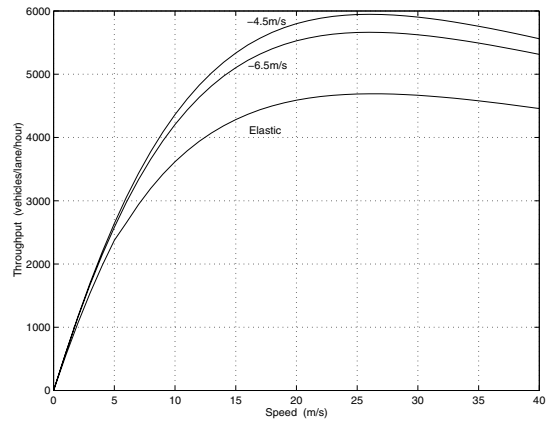


Figure 33: Variation of throughput with Elasticity

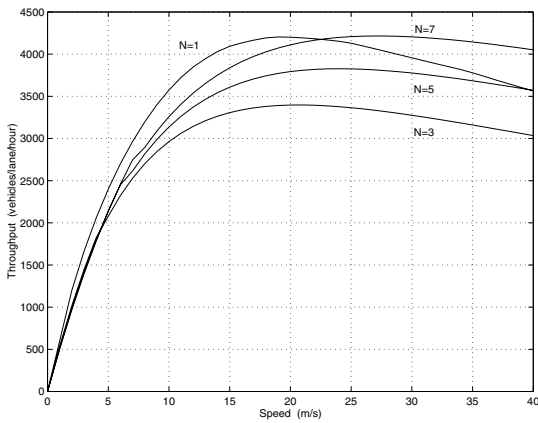


Figure 31: Throughput with large width braking distribution: own capability known, minimum headway 0.5s

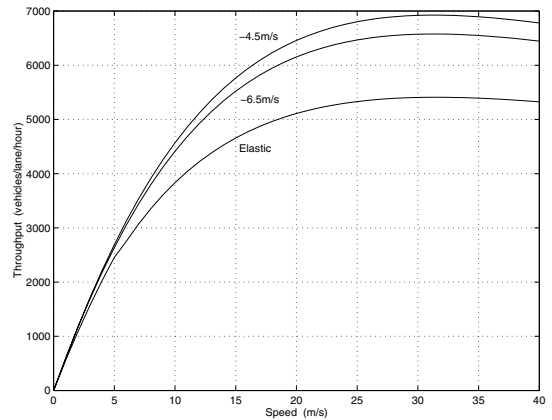


Figure 34: Variation of throughput with elasticity: own capability known

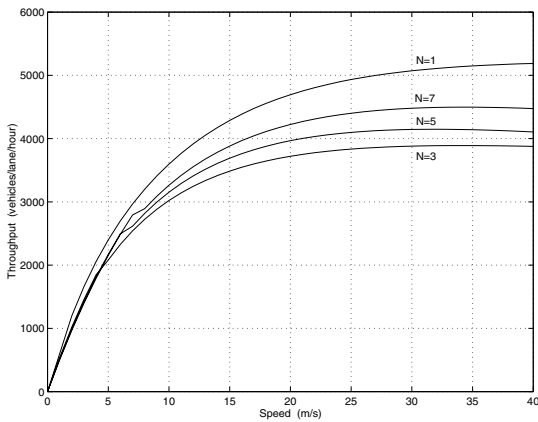


Figure 32: Throughput with large width braking distribution: both capabilities known, minimum headway 0.5s

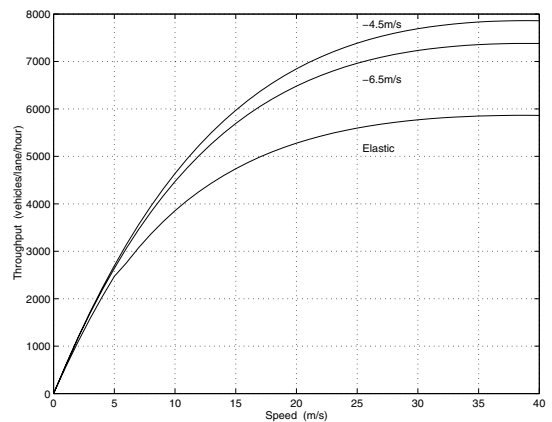


Figure 35: Variation of throughput with elasticity: both capabilities known

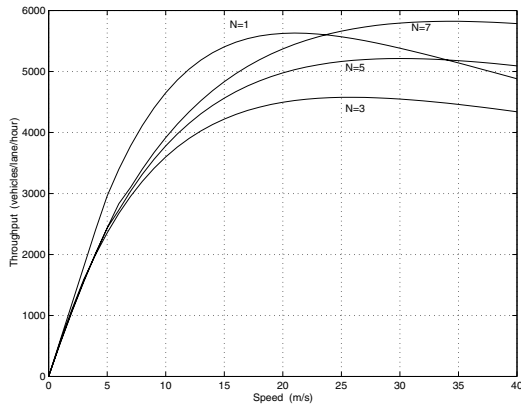


Figure 36: Capability estimation error = 1 bin-width: own capability known

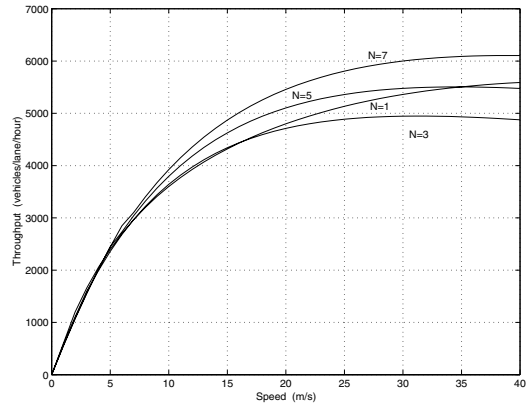


Figure 39: Capability estimation error = 1 bin-width: both capabilities known, min. headway 0.5s

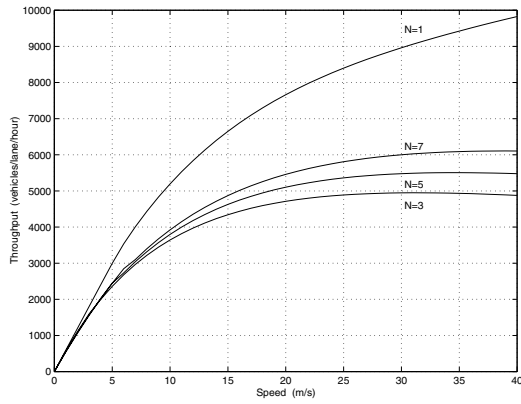


Figure 37: Capability estimation error = 1 bin-width: both capabilities known

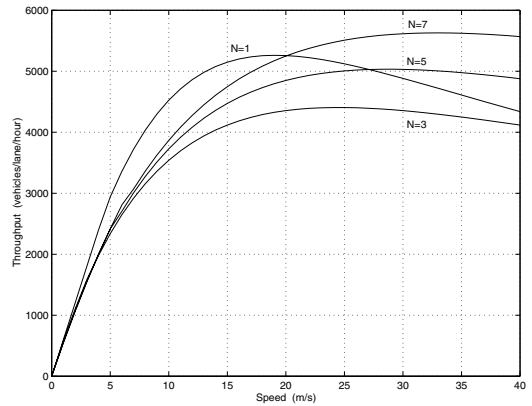


Figure 40: Capability estimation error = 2 bin-width: own capability known

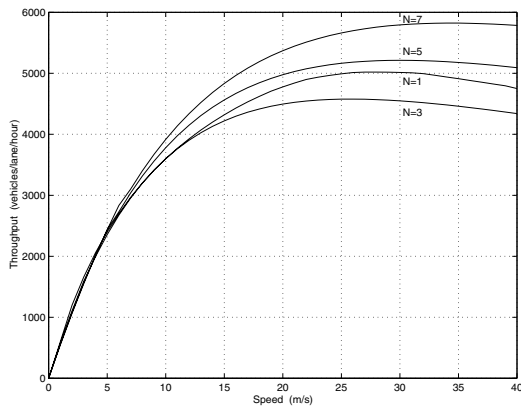


Figure 38: Capability estimation error = 1 bin-width: own capability known, min. headway 0.5s

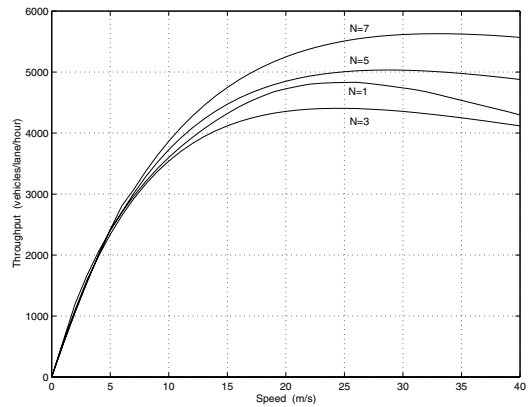


Figure 41: Capability estimation error = 2 bin-width: own capability known, min. headway 0.5s

C Vehicle models to capture different AHS designs

C.1 Vehicle following with Delays

We model the longitudinal dynamics of a vehicle by a second order system along with a first order lag on the input side representing lumped sensor, actuator and processing delays. Thus, the dynamics of vehicle A is represented by

$$\begin{aligned}\ddot{x}_A &= u_{actual} \\ \dot{u}_{actual} &= \frac{1}{\tau}u - \frac{1}{\tau}u_{actual}\end{aligned}$$

where u is the input applied by the controller and τ represents the lumped lag. The acceleration of vehicle B is still treated as an unmeasured disturbance. The dynamics of the combined A-B system can be characterized by the state vector $x = [\dot{x}_A, \ddot{x}_A, D_{AB}, \dot{D}_{AB}]^T$ where D_{AB} represents the distance between the front bumper of A and rear bumper of B as before. The system state evolves according to the following equation:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{\tau} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{\tau} \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \ddot{x}_B := Ax + Bu + E\ddot{x}_B$$

The constraints on the dynamics are:

$$\begin{aligned}x(t) \in \{x \in \mathbb{R}^4 | x_1 \in [v_A^{min}, v_A^{max}], x_2 \in [a_A^{min}, a_A^{max}], x_4 + x_1 \in [v_B^{min}, v_B^{min}]\} \\ \ddot{x}_A \in [j_A^{min}, j_A^{max}] \quad \text{and} \quad \ddot{x}_B(t) \in [a_B^{min}, a_B^{max}]\end{aligned}$$

The collisions are modeled as before. Thus, the effect of vehicles B and C on vehicle A can be summarized as a disturbance given by equation (2). By using different values for the lag τ , we can model varying degrees of cooperation between vehicles, ranging from no inter-vehicle cooperation to emergency warning.

Consider the candidate saddle strategy:

$$u^*(t) = \begin{cases} x_2^0 - |j_A^{min}|t & \text{if } t \leq T_1 \\ a_A^{min} & \text{if } t > T_1 \end{cases} \quad (19)$$

$$d^*(t) = \{\ddot{x}_B^*, (T_B^*, \delta v_B^*), (T_C^*, \delta v_C^*)\} \quad (20)$$

where $T_B^* = T_C^* = 0$, $\delta v_B^* = \max\{v_B, x_4^0 + x_1^0\}$, $\delta v_C^* = v_C$ and:

$$\ddot{x}_B^*(t) = \begin{cases} a_B^{min} & \text{if } t \leq T_2 \\ 0 & \text{if } t > T_2 \end{cases} \quad (21)$$

T_1 is the time when the acceleration of vehicle A, $x_2(T_1)$, reaches a_A^{min} under the control law $u^*(t)$ given above and T_2 is the time when vehicle B stops under a_B^{min} . The candidate saddle strategy is such that vehicle B applies maximum acceleration and both the vehicles B & C collide at $t = 0$ with the maximum permissible speeds. The control $u^*(t)$ consists of commanding the maximum braking at the maximum possible rate. Note that due to the lag time constant τ , the commanded acceleration can not be exactly followed. We assume that the actual lag is unknown and is bounded above by τ . If the lag model is exactly known, then one can compensate for it so that vehicle A will follow the trapezoidal acceleration trajectory (given by the acceleration and jerk constraints) closely.

Under the current assumptions, the controller $u^*(t)$ is the optimal controller that also guarantees that all the constraints are satisfied.

The dynamical equations can be explicitly integrated by using the formula:

$$x(t) = e^{At} x^0 + \int_0^t e^{A(t-\delta)} B u(\delta) d\delta + \int_0^t e^{A(t-\delta)} E d(\delta) d\delta$$

As the matrix A is nilpotent, e^{At} can be explicitly computed

$$e^{At} = \begin{bmatrix} 1 & \tau(1 - e^{-\frac{t}{\tau}}) & 0 & 0 \\ 0 & e^{-\frac{t}{\tau}} & 0 & 0 \\ 0 & \tau^2(1 - e^{-\frac{t}{\tau}}) - t\tau & 1 & t \\ 0 & \tau(e^{-\frac{t}{\tau}} - 1) & 0 & 1 \end{bmatrix}$$

Following the steps in [13], it can be shown that,

Lemma 3 (u^*, d^*) is a global saddle solution for cost $J(x^0, u, d)$.

Substituting $(u^*(t), d^*(t))$ in the system equations yield the optimum trajectory $x^*(t)$. Let T_3 be the stopping time for vehicle A. Note that $x_3^*(t)$ is a differentiable function of t on a compact interval $[0, T_3]$, with derivative $\dot{x}_3^*(t)$. Therefore, the distance between vehicles A and B will be minimum under the optimum control either at $t = 0$ or at $t = T_3$ or at $\hat{T} \in [0, T_3]$, s.t. $x_4^*(\hat{T}) = 0$. The computer program thus calculates $J^*(x^0, u^*, d^*) = -x_3^*(t)$ for a given x^0 . This allows us to calculate the boundary separating safe initial conditions from unsafe ones, thereby providing the minimum safe distance between vehicles A and B.

C.2 Vehicle following with communication (no sensing delay)

In this case, we allow each vehicle (platoon) to communicate its own acceleration to the following vehicle (platoon)⁹. To compare this situation with that of no communication, we assume that the communication delay is negligible. Thus we model vehicles A and B as triple integrators with the jerk of vehicle A being the control variable and the jerk of vehicle B acting as the continuous disturbance. The interaction between vehicles A and B can be captured by the state vector $x = [\dot{x}_A \ \ddot{x}_A \ D_{AB} \ \dot{D}_{AB} \ \ddot{D}_{AB}]^T$. The system dynamics are given by:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \ddot{x}_B := Ax + Bu + E\ddot{x}_B$$

The constraints on the dynamics are given by:

$$x(t) \in \{x \in \mathbb{R}^4 | x_1 \in [v_A^{min}, v_A^{max}], x_2 \in [a_A^{min}, a_A^{max}], x_4 + x_1 \in [v_B^{min}, v_B^{max}], x_5 + x_2 \in [a_B^{min}, a_B^{max}]\} \\ u(t) \in [j_A^{min}, j_A^{max}] \quad \text{and} \quad \ddot{x}_B(t) \in [j_B^{min}, j_B^{max}]$$

The collisions are modeled as before. Thus, the effect of vehicles B and C on vehicle A can be summarized as a disturbance given by equation (2).

⁹Note that acceleration information is necessary for followers within a platoon. Here we are considering the inter-platoon information exchange.

In this case, every vehicle has information about the acceleration of the preceding vehicle. We consider the following candidate strategy:

$$u^*(t) = \begin{cases} j_A^{min} & \text{if } t \leq T_1 \\ 0 & \text{if } t > T_1 \end{cases} \quad (22)$$

$$d^*(t) = \{\ddot{x}_B^*, (T_B^*, \delta v_B^*), (T_C^*, \delta v_C^*)\} \quad (23)$$

where $T_B^* = T_C^* = 0$, $\delta v_B^* = \max\{v_B, x_4^0 + x_1^0\}$, $\delta v_C^* = v_C$ and:

$$\ddot{x}_B^*(t) = \begin{cases} j_B^{min} & \text{if } t \leq T_2 \\ 0 & \text{if } t > T_2 \end{cases} \quad (24)$$

T_1 is the time when the acceleration of vehicle A reaches a_A^{min} under jerk j_A^{min} , and T_2 is the time when the acceleration of vehicle B reaches its minimum a_B^{min} under jerk j_B^{min} . The candidate saddle strategy is such that both vehicles A and B apply maximum braking at the maximum possible rate at $t = 0$, and the vehicles B & C collide at $t = 0$ with the maximum permissible speeds.

The dynamical equations can be explicitly integrated by using the formula:

$$x(t) = e^{At} x^0 + \int_0^t e^{A(t-\delta)} B u(\delta) d\delta + \int_0^t e^{A(t-\delta)} E d(\delta) d\delta$$

As the matrix A is nilpotent, e^{At} can be explicitly computed

$$e^{At} = \begin{bmatrix} 1 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Following the steps in [13], it can be shown that,

Lemma 4 (u^*, d^*) is a global saddle solution for cost $J(x^0, u, d)$.

Similar to the previous case, the computer program calculates $J^*(x^0, u^*, d^*) = -x_3^*(t)$ for a given x^0 which allows us to calculate the boundary separating safe initial conditions from unsafe ones. This results in the the minimum safe distance between vehicles A and B as a function of initial conditions and system parameters.

C.3 Vehicle following with communication (with sensing delay)

Here, we model the vehicles of Section C.1 with the added feature that a vehicle communicates its acceleration to the following vehicle. The commanded acceleration of vehicle B acts as a continuous disturbance for vehicle A. The sensing and actuation lags (modeled by parameter τ) are considered to be the same for vehicles A & B. With the state vector given by $x = [\dot{x}_A \ \ddot{x}_A \ D_{AB} \ \dot{D}_{AB} \ \ddot{D}_{AB}]^T$, the system dynamics are given by:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\tau} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\tau} & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{\tau} \\ 0 \\ 0 \\ -\frac{1}{\tau} \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \ddot{x}_B := Ax + Bu + E\ddot{x}_B$$

The constraints on the dynamics are given by:

$$x(t) \in \{x \in \mathbb{R}^4 | x_1 \in [v_A^{min}, v_A^{max}], x_2 \in [a_A^{min}, a_A^{max}], x_4 + x_1 \in [v_B^{min}, v_B^{max}], x_5 + x_2 \in [a_B^{min}, a_B^{max}]\}$$

$$\ddot{x}_A(t) \in [j_A^{min}, j_A^{max}] \quad \text{and} \quad \ddot{x}_B(t) \in [j_B^{min}, j_B^{max}]$$

The collisions are modeled as before. The effect of vehicles B and C on vehicle A is modeled as a disturbance given by equation (2).

Consider the candidate strategy:

$$u^*(t) = \begin{cases} x_2^0 - |j_A^{min}|t & \text{if } t \leq T_1 \\ a_A^{min} & \text{if } t > T_1 \end{cases} \quad (25)$$

$$d^*(t) = \{\ddot{x}_B^*, (T_B^*, \delta v_B^*), (T_C^*, \delta v_C^*)\} \quad (26)$$

where $T_B^* = T_C^* = 0$, $\delta v_B^* = \max\{v_B, x_4^0 + x_1^0\}$, $\delta v_C^* = v_C$ and:

$$\ddot{x}_B^*(t) = \begin{cases} j_B^{min} & \text{if } t \leq T_2 \\ 0 & \text{if } t > T_2 \end{cases} \quad (27)$$

T_1 is the time when the acceleration of vehicle A, $x_2(T_1)$, reaches a_A^{min} under the control law $u^*(t)$ given above and T_2 is the time when the acceleration of vehicle B reaches a_B^{min} under application of maximum jerk j_B^{min} . The candidate saddle strategy is such that the vehicle B applies maximum jerk until it reaches maximum acceleration level and both the vehicles B & C collide at $t = 0$ with the maximum permissible speeds. The control $u^*(t)$ consists of commanding the maximum braking at the maximum possible rate while maintaining system constraints.

The dynamical equations can be explicitly integrated by using the formula:

$$x(t) = e^{At} x^0 + \int_0^t e^{A(t-\delta)} B u(\delta) d\delta + \int_0^t e^{A(t-\delta)} E d(\delta) d\delta$$

As the matrix A is nilpotent, e^{At} can be explicitly computed

$$e^{At} = \begin{bmatrix} 1 & \tau(1 - e^{-\frac{t}{\tau}}) & 0 & 0 & 0 \\ 0 & e^{-\frac{t}{\tau}} & 0 & 0 & 0 \\ 0 & \tau^2(1 - e^{-\frac{t}{\tau}}) + \frac{t^2}{2} - t\tau & 1 & t & \frac{t^2}{2} \\ 0 & t + \tau(e^{-\frac{t}{\tau}} - 1) & 0 & 1 & t \\ 0 & (1 - e^{-\frac{t}{\tau}}) & 0 & 0 & 1 \end{bmatrix}$$

Following the steps in [13], it can be shown that,

Lemma 5 (u^*, d^*) is a global saddle solution for cost $J(x^0, u, d)$.

Substituting $(u^*(t), d^*(t))$ in the system equations yield the optimum trajectory $x^*(t)$. The computer program similarly calculates $J^*(x^0, u^*, d^*) = -x_3^*(t)$ for a given x^0 in order to calculate minimum safe spacing between vehicles.